Robust Utility Maximization, f-Projections, and Risk Constraints

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Abstract

Finding payoff profiles that maximize the expected utility of an agent under some budget constraint is a key issue in financial mathematics. We characterize optimal contingent claims for an agent who is uncertain about the market model. The dual approach that we use leads to a minimization problem for a certain convex functional over two sets of measures, which we first have to solve. Finally, we incorporate a second constraint that limits the risk that the agent is allowed to take. We will proceed as follows:

Chapter 1

Given a convex function f and two sets $\mathcal{P}$ and $\mathcal{Q}$ of probability measures, we consider the problem of minimizing the robust f-divergence $\inf_{Q\in\mathcal{Q}} f(P|Q)$ over $P \in \mathcal{P}$. We show that, if $\mathcal{P}$ is closed and $\mathcal{Q}$ weakly compact, a minimizer exists within the class $\mathcal{P}$ if $\lim_{x\to\infty} f(x)/x = \infty$. The key step is to prove that a certain relevant subset of $\mathcal{P}$ is weakly compact. To this end, we use Young's inequality in an appropriate Orlicz space. $\mathcal{P}$ may be interpreted as the set of martingale measures for some semimartingale. Furthermore, we show that if $\mathcal{Q}$ is weakly compact and $\lim_{x\to\infty} f(x)/x = 0$, then there is a minimizer in a class $\bar{\mathcal{P}}$ of extended martingale measures defined on the predictable σ-field.

Chapter 2

The existence results in Chapter 1 lead to the existence of a contingent claim which maximizes the robust utility functional $\inf_{Q\in\mathcal{Q}} E_Q[u(X)]$ over some set of affordable contingent claims. Such a utility functional satisfies the axioms of Gilboa and Schmeidler and can be interpreted as the utility of an agent who is uncertain about the market model Q, and who therefore considers a whole set $\mathcal{Q}$ of subjective or model measures. In order to solve the problem of maximizing this robust utility functional, we distinguish between utility functions that are finite on the whole real line and utility functions that are only defined on the positive halfline. These two cases correspond to the different existence results from the first chapter. The key idea is to identify the minimizing measures P^* and Q^* as certain worst case measures. Then we are able to reduce the robust problem in an incomplete market to the classical problem of maximizing the expected utility $E_{Q^*}[u(X)]$ under a cost constraint in terms of P^*.

Chapter 3

Finally, we incorporate an additional constraint: We require the risk of the contingent claims to be bounded, where we define risk in terms of utility-based shortfall risk. We first give a solution to the resulting optimization problem for a classical utility functional in a complete market model. Then we solve the corresponding robust problem in an incomplete market for a utility function that is only defined on the positive halfline. Here we use a generalized duality approach. In an example we compare the optimal claim under this risk constraint with the optimal claims without a risk constraint and under a value-at-risk constraint.

Keywords:
utility maximization, model uncertainty, f-divergences, risk constraints

Zusammenfassung

Ein wichtiges Gebiet der Finanzmathematik ist die Bestimmung von Auszahlungsprofilen, die den erwarteten Nutzen eines Agenten unter einer Budgetrestriktion maximieren. Wir charakterisieren optimale Auszahlungsprofile für einen Agenten, der unsicher ist in Bezug auf das genaue Marktmodell. Der hier benutzte Dualitätsansatz führt zu einem Minimierungsproblem für bestimmte konvexe Funktionale über zwei Mengen von Wahrscheinlichkeitsmaßen, das wir zunächst lösen müssen. Schließlich führen wir noch eine zweite Restriktion ein, die das Risiko beschränkt, das der Agent eingehen darf. Wir gehen dabei wie folgt vor:

Kapitel 1

Wir betrachten das Problem, die robuste f-Divergenz $\inf_{Q\in\mathcal{Q}} f(P|Q)$ über $P \in \mathcal{P}$ zu minimieren, wobei f eine konvexe Funktion und $\mathcal{P}$ und $\mathcal{Q}$ zwei Mengen von Wahrscheinlichkeitsmaßen sind. Wir zeigen, dass unter der Bedingung $\lim_{x\to\infty} f(x)/x = \infty$ ein Minimierer in der Menge $\mathcal{P}$ existiert, falls $\mathcal{P}$ abgeschlossen und $\mathcal{Q}$ schwach kompakt sind. Entscheidend ist hierbei der Beweis der schwachen Kompaktheit einer bestimmten relevanten Teilmenge von $\mathcal{P}$. Dazu benutzen wir die Young-Ungleichung auf einem geeigneten Orlicz-Raum. Unter $\mathcal{P}$ kann man sich die Menge der Martingalmaße für ein Semimartingal vorstellen. Außerdem zeigen wir, dass unter der Bedingung $\lim_{x\to\infty} f(x)/x = 0$ ein Minimierer in einer erweiterten Klasse von Martingalmaßen existiert, falls $\mathcal{Q}$ schwach kompakt ist. Diese erweiterte Klasse wird auf der σ-Algebra der prävisiblen Ereignisse definiert.

Kapitel 2

Die Existenzresultate aus dem ersten Kapitel implizieren die Existenz eines Auszahlungs-profils, das das robuste Nutzenfunktional $\inf_{Q \in \mathcal{Q}} E_Q[u(X)]$ über eine Menge von finanzierbaren Aus-zahlungen maximiert. Solch ein Nutzenfunktional erfüllt die Axiome von Gilboa und Schmeidler, und es kann als Nutzen eines Agenten interpretiert werden, der das Markmodell Q nicht genau kennt und deshalb eine ganze Menge $\mathcal{Q}$ von solchen subjektiven Modellmaßen betrachtet. Um das robuste Nutzenmaximierungsproblem zu lösen, unterscheiden wir zwischen Nutzenfunktionen, die auf der ganzen reellen Achse endlich sind, und solchen, die nur auf der positiven Halbachse definiert sind. Diese beiden Fälle entsprechen den verschiedenen Existenzresultaten im ersten Kapitel. Die entscheidende Idee besteht darin, die minimierenden Maße P^* und Q^* als gewisse "worst-case"-Maße zu identifizieren. Damit ist es möglich, das robuste Problem in einem unvollständigen Markt auf ein klassisches Problem zu reduzieren, in dem man den erwarteten Nutzen $E_{Q^*}[u(X)]$ unter einer Budgetrestriktion, die nur mit Hilfe von P^* definiert wird, maximiert.

Kapitel 3

Schließlich führen wir eine zusätzliche Restriktion ein: Wir fordern, dass das Risiko der Auszahlungsprofile beschränkt ist, wobei wir Risiko mittels "utility-based shortfall risk" definieren. Zunächst betrachten wir das resultierende Optimierungsproblem für ein klassisches Nutzenfunktional in einem vollständigen Marktmodell. Dann lösen wir das entsprechende robuste Problem in einem unvollständigen Marktmodell für Nutzenfunktionen, die nur auf der positiven Halbachse definiert sind. In einem Beispiel vergleichen wir das optimale Auszahlungsprofil unter der Risikorestriktion mit den optimalen Auszahlungen ohne eine solche Restriktion und unter einer Value-at-Risk-Nebenbedingung.

Schlagwörter:
Nutzenmaximierung, Modellunsicherheit, f-Divergenzen,
Risiko-Nebenbedingung

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Chapter 0

Introduction

A fundamental problem in financial mathematics is the characterization of investments that are optimal given an agent's preferences and his budget constraint. In recent years, model uncertainty has become a topic of interest, and the maximization of robust utility functionals under model uncertainty has been considered in several papers. In this thesis I characterize the solution to such a robust utility maximization problem in an incomplete market in terms of the density of certain measures. These measures solve a projection problem which consists of minimizing a certain f-divergence over the two sets of martingale measures and subjective model measures. In the first chapter of this thesis, we show that under suitable assumptions there is a solution to this projection problem. Using this existence result, we then present the solution to the robust utility maximization problem in the second chapter. A new class of utility maximization problems arises by including a second constraint in addition to the budget constraint: We limit the risk that an investor is allowed to take. The solution to such a problem is the focus of the last chapter of the thesis.

Robust Utility Maximization and the Dual Problem

We consider an agent who wants to determine a payoff profile or contingent claim in a financial market that is optimal with respect to his preferences. Typically, such preferences admit a numerical representation U, and under suitable assumptions U can be described by means of an expectation. Von Neumann & Morgenstern [1944] and Savage [1954] formulated axioms under which the utility of a contingent claim can be expressed in terms of a utility function u and a probability measure Q, i.e.,

$$U(X) = E_Q[u(X)].$$

However, both from a normative and descriptive point of view, there are good reasons to consider alternative utility functionals. Gilboa and Schmeidler [1989] proposed a more flexible set of axioms for preference orders on payoff profiles. It led to a numerical representation by a *robust utility functional* of the form

$$U(X) = \inf_{Q \in \mathcal{Q}} E_Q[u(X)], \tag{1}$$

where u is again a utility function, and $\mathcal{Q}$ is a set of *model* or *subjective measures*. This approach covers the uncertainty of the probability of market events: The agent considers a whole set $\mathcal{Q}$ of possible models and takes a worst case approach in evaluating the expected utility of a payoff. See the book by Föllmer and Schied [2004] for an overview of the theory of preference orders and numerical representations.

Our aim is now to determine a contingent claim or payoff profile X^* that maximizes the *robust utility functional* (1). Let us consider a financial market which is modelled by a filtered probability space with a semimartingale representing the price processes of the stocks in the market. Denote by $\mathcal{P}_e$ the set of *equivalent local martingale measures* for this semimartingale, which is assumed to be non-empty in order to exclude arbitrage. When maximizing his expected utility, the agent is assumed to own a certain amount of capital x_0 that he is allowed to spend. The price of the considered contingent claims then has to be bounded by the initial endowment x_0. If the financial market is assumed to be *complete*, every contingent claim is attainable by some self-financing trading strategy, and thus the arbitrage-free price of a contingent claim is determined by the expectation under the unique equivalent martingale measure P. Under the classical von-Neumann-Morgenstern or Savage axioms, the problem of finding an optimal payoff profile can therefore be formulated as

$$\text{Maximize } E_Q[u(X)] \text{ over all contingent claims } X \text{ that satisfy } E_P[X] \le x_0. \tag{2}$$

For a continuous-time model, this problem was first studied by Merton [1969] and [1971]. Assuming that the stock price process is Markovian and using methods of stochastic control, he obtained solutions for the power, logarithmic, and exponential utility functions. Another method of solving this problem is the martingale or duality approach, where the assumption of a Markovian stock price process can be dropped. This approach was developed by Pliska [1986], Karatzas et al. [1987], and Cox and Huang [1989] and [1991] for complete market models. For an overview and very clear explanations of this duality theory, see the lecture by Rogers [2003] and the book by Karatzas and Shreve [1998].

In an *incomplete* financial market the optimization problem becomes more challenging: contingent claims are in general not attainable by self-financing trading strategies, and instead of a single equivalent martingale measure there is a whole set $\mathcal{P}_e$ of such measures. This leads to infinitely many prices that are consistent with the absence of arbitrage. Instead of the attainability of a contingent claim, one usually requires the existence of a super-replicating strategy, that is, a strategy whose corresponding value process dominates the payoff of our contingent claim. If the contingent claim is bounded from below, then due to the optional decomposition theorem by Kramkov [1996] this is satisfied if and only if the superhedging price $\sup_{P\in\mathcal{P}_e} E_P[X]$ is bounded by the initial endowment x_0. This in turn is equivalent to

$$\sup_{P\in\mathcal{P}} E_P[X] \leq x_0, \tag{3}$$

where $\mathcal{P}$ is the set of *absolutely continuous martingale measures*. Under the von-Neumann-Morgenstern or Savage axioms the problem of finding an optimal payoff profile can then be formulated as

$$\text{Maximize } E_Q[u(X)] \text{ over all contingent claims } X$$
$$\text{that satisfy } \sup_{P\in\mathcal{P}} E_P[X] \leq x_0.$$

A solution to this problem was obtained by He and Pearson [1991b] for a discrete-time model on a finite probability space. He and Pearson [1991a] and Karatzas et al. [1991] studied the problem in a continuous-time diffusion model. In a general semimartingale model, this problem was solved by Frittelli [2000] for the exponential utility function, by Kramkov and Schachermayer [1999] and [2003] for utility functions which are defined on the positive halfline, by Goll and Rüschendorf [2001] for general utility functions, and by Bellini and Frittelli [2002] and by Schachermayer [2001] for utility functions which are finite on the whole real line. In the case where the utility function is finite on the whole real line, the solution to the utility maximization problem is in general not bounded from below, and instead of the constraint (3) we can only require $E_P[X]$ to be bounded by x_0 for P in a certain subset of $\mathcal{P}$.

In this thesis we want to consider the *robust utility maximization problem* under the Gilboa-Schmeidler axioms in an incomplete market, i.e.,

$$\begin{aligned} &\text{Maximize } \inf_{Q\in\mathcal{Q}} E_Q[u(X)] \text{ over all contingent claims } X \\ &\qquad \text{that satisfy } \sup_{P\in\mathcal{P}} E_P[X] \le x_0. \end{aligned} \tag{4}$$

Baudoin [2002] solved such a problem for the special case of a complete market model of "weak information", which means that $\mathcal{Q}$ is the set of measures under which some given random variable has a specific law. Schied [2005b] solved Problem (4) in a complete market model for utility functions that are defined on the positive halfline. Under an L^p-integrability condition on the set of subjective measures $\mathcal{Q}$, Quenez [2004] obtained a solution for utility functions on the positive halfline and for equivalent subjective measures. Recently, Schied and Wu [2005] solved Problem (4) without the equivalence assumption on $\mathcal{Q}$ for utility functions on the positive halfline. Burgert and Rüschendorf [2005] considered a robust utility maximization problem where the utility is obtained from consumption. The axioms of preference orders by Gilboa and Schmeidler [1989] can be relaxed further, and this was done by Maccheroni et al. [2004]. Schied [2005a] solved the corresponding utility maximization problem.

In all of the above articles dealing with the robust utility maximization problem, a martingale or duality approach was used. In the general case of Problem (4), the *dual problem* consists of minimizing a certain *f-divergence* over the sets of martingale and subjective measures. For a convex function f, the f-divergence of a measure P with respect to Q is given by

$$f(P|Q) := E_Q\left[f\left(\frac{dP^a}{dQ}\right)\right] + \lim_{x\to\infty}\frac{f(x)}{x}P^s(\Omega),$$

where P^a and P^s denote the absolutely continuous and singular part in the Hahn-Lebesgue decomposition of P with respect to Q. Common examples for f-divergences are the relative entropy with $f(x) = x\log x$ and the reverse relative entropy with $f(x) = -\log x$. Due to Jensen's inequality, the f-divergence may be considered as a measure of distance between probability distributions. We call a measure P that minimizes the f-divergence with respect to Q over the set $\mathcal{P}$ an *f-projection* of Q on $\mathcal{P}$, and a measure that minimizes the *robust f-divergence* $\inf_{Q\in\mathcal{Q}} f(P|Q)$, a *robust f-projection* of $\mathcal{Q}$ on $\mathcal{P}$. In the context of utility maximization we set $f(x) := v(\lambda x)$ for some $\lambda > 0$, where v is the convex conjugate of the utility function u. The dual problem then is to

$$\text{Minimize } \inf_{Q\in\mathcal{Q}} f(P|Q) \text{ over } P \in \mathcal{P}. \tag{5}$$

When solving this problem we have to distinguish between two types of utility functions: If the utility function is finite on the whole real line, then $\lim_{x\to\infty} f(x)/x = \infty$ for the corresponding convex function f in the dual problem. If the utility function is only defined for positive values, then $\lim_{x\to\infty} f(x)/x = 0$.

In the case where the set $\mathcal{Q}$ reduces to a singleton and if $\lim_{x\to\infty} f(x)/x = \infty$, existence of a solution to Problem (5) was shown by Csiszár [1975] for the case of relative entropy, and by Liese and Vajda [1987] and Bellini and Frittelli [2002] for general convex functions f. Rüschendorf [1984] gave very useful characterizations of the f-projection which can be applied when solving the utility maximization problem. If the set $\mathcal{Q}$ consists of more than one measure, Csiszár and Tusnády [1984] obtained existence results for robust projections in two special cases: (i) for the relative entropy on a finite space, and (ii) for the squared L^2-distance between the densities of P and Q.

In Chapter 1 we analyze the robust projection problem in its general form (5). Our main result in Section 1.1 is Theorem 1.1.2. It states that a solution exists if

$$\lim_{x\to\infty} \frac{f(x)}{x} = \infty, \tag{6}$$

the set $\mathcal{P}$ is closed in variation, and the set $\mathcal{Q}$ is weakly compact. The key step is to show that $\{f(\cdot|\cdot) \leq c\}$, viewed as a subset of $L^1(\mathbb{R}) \times L^1(\mathbb{R})$, is weakly compact. In the classical case with $\mathcal{Q} = \{Q_0\}$ this follows easily from (6) using the de la Vallée-Poussin compactness criterion. In the general robust case the proof is more delicate. Instead of applying the compactness criterion in terms of f, we have to construct an auxiliary convex function l satisfying (6) such that the compactness condition in terms of l follows via Young's inequality in an appropriate Orlicz space.

For utility functions on the positive halfline, Kramkov and Schachermayer [1999] showed how to develop the duality between the classical problem of utility maximization with $\mathcal{Q} = \{Q_0\}$ and the projection problem (5) beyond the class $\mathcal{P}$: A martingale measure P is identified with the martingale of its density process with respect to the reference measure R, this class of martingales is embedded in a suitable class of supermartingales, and the projection problem is solved within this larger class. Cvitanic et al. [2001] showed how to describe the solution of the projection problem as a finitely additive measure. Recently Quenez [2004] and Schied and Wu [2005] extended the solution by Kramkov and Schachermayer [1999] to the robust case.

In Section 1.2 we consider the robust projection problem in the case

$$\lim_{x\to\infty} \frac{f(x)}{x} = 0,$$

which can be interpreted as the dual problem for a utility function that is only defined on the positive halfline. Instead of taking a set of supermartingales as introduced by Kramkov and Schachermayer [1999], we insist on the original idea of identifying a solution to the robust projection problem (5) within a suitable class of martingale measures. As shown by Föllmer [1972] and [1973], any supermartingale on a sufficiently rich filtered probability space can be represented as a probability measure on the predictable σ-field. For such measures, we introduce the notion of an *extended martingale measure.* Theorem 1.2.8 shows how the robust projection problem can be solved in the class $\bar{\mathcal{P}}$ of extended martingale measures. Some of the key arguments are essentially the same as in Quenez [2004] and Schied and Wu [2005]. The main novelty is that here we insist on an appropriate notion of a martingale measure.

In Chapter 2 we then show how the existence of such minimizing measures leads to the solution of the robust utility maximization problem (4). Our main result is presented in Theorems 2.3.9 and 2.3.10, where we solve Problem (4) for the two cases of utility functions, respectively. It is given by

$$X^* := I\left(\lambda^* \frac{dP^*}{dQ^*}\right), \tag{7}$$

where $I := (u')^{-1}$, λ^* is some suitable Lagrange multiplier, P^* is the solution to the dual problem (5) with $f(x) := v(\lambda^* x)$, and Q^* is its reverse f-projection, that is, it minimizes the f-divergence $f(P^*|Q)$ over the set $\mathcal{Q}$. This is proven using a duality approach and the existence result from the first chapter. We first solve the simplified problem (2) in a slightly generalized form, where we do not assume that the martingale measure P is absolutely continuous with respect to the model measure Q as it was in the above-mentioned papers. In order to solve the robust problem (4) in an incomplete market, we then characterize the measures P^* and Q^* as worst case measures for the robust utility maximization problem within their respective sets. This result goes back to Theorem 5 by Rüschendorf [1984], and it is given in Proposition 2.3.8. It allows us to reduce the robust problem (4) in an incomplete market to the classical problem (2) under the reverse f-projection Q^* and with the robust f-projection P^* as prizing measure. This finally leads to the solution (7). To the best of our knowledge, we are the first to solve this robust utility maximization problem for the case of utility functions that are finite on the whole real line. For utility functions that are only defined on the positive halfline, this problem was solved by Quenez [2004] under an L^p-integrability and an equivalence assumption on the set of subjective measures $\mathcal{Q}$. Schied and Wu [2005] removed the two assumptions and replaced them by a compactness criterion for the set $\mathcal{Q}$. We work under similar assumptions, but as in the projection problem we insist on the representation of the solution (7) in terms of the density of the extended martingale measure P^* with respect to the subjective measures Q^*.

Parts of Chapter 2 have already appeared in Gundel [2005], but here we are able to relax the equivalence and integrability assumptions used there. The results in Chapter 1 together with parts of Chapter 2 will also appear in a joint paper with Hans Föllmer [2006].

The Utility Maximization Problem Under an Additional Risk Constraint

In the last five years the utility maximization problem has been linked to the discussion of risk measures. By introducing coherent risk measures, Artzner et al. [1999] brought about an intense research on the topic of suitable representations of the risk of an investment. Föllmer and Schied [2002a] and [2002b] and Frittelli and Rosazza Gianin [2002] generalized the assumptions of Artzner et al. [1999] by introducing convex risk measures. For an overview of this topic, we refer to the book by Föllmer and Schied [2004].

When maximizing the expected utility, in addition to the requirement (3) of affordability a second constraint has been incorporated in the utility maximization literature: the limitation of the risk of the investment. Regulators, for example, might impose a risk constraint to certain companies, or a manager of a firm might require his traders to stay within some risk limit. Basak and Shapiro [2001] examined the utility maximization problem under a joint budget and risk constraint for the case where risk is defined in terms of value at risk and in terms of expected loss. They considered expected loss defined by the expectation $E_P[(X-q)^-]$ for some $q \in \mathbb{R}$, where P is the unique equivalent martingale measure in a complete market model. They gave a characterization of the solution if both constraints are binding, but neither showed in which situations the constraints are indeed binding, nor proved the existence of a solution. Gabih et al. [2005a] solved the problem with expected loss being defined as $E_Q[(X-q)^-]$, and Gabih et al. [2005b] generalized this to loss functionals of the type $E_Q[(u(X)-u(X_0))^-]$, where X_0 might be stochastic. Here we want to consider a constraint defined in terms of *utility-based shortfall risk.* For a convex loss function l and a subjective measure Q_1, we define an acceptance set by

$$\mathcal{A}_{Q_1} = \{X : \ E_{Q_1}[\ell(-X)] \leq x_1\}$$

for some subjective measure Q_1 and a threshold x_1. That is, X is acceptable if its expected loss under Q_1 is smaller than x_1. Utility-based shortfall risk is the corresponding convex risk measure

$$\rho_{Q_1}(X) = \inf\{m \in \mathbb{R} : \ X + m \in \mathcal{A}_{Q_1}\}.$$

For a detailed description of the properties of utility-based shortfall risk, we refer to Weber [2005], Dunkel and Weber [2005], and Giesecke et al. [2005].

Our aim in Chapter 3 is to solve the utility maximization problem under both a budget and a risk constraint under model uncertainty, i.e.,

$$\begin{gathered} \text{Maximize } \inf_{Q \in \mathcal{Q}} E_Q[u(X)] \text{ over all contingent claims } X \\ \text{that satisfy } \sup_{P \in \mathcal{P}} E_P[X] \leq x_0 \text{ and } \sup_{Q_1 \in \mathcal{Q}_1} \rho_{Q_1}(X) \leq 0. \end{gathered} \tag{8}$$

As in Chapter 2, we first consider the simplified problem in Section 3.2, i.e.,

$$\begin{gathered} \text{Maximize } E_Q[u(X)] \text{ over all contingent claims } X \\ \text{that satisfy } E_P[X] \leq x_0 \text{ and } \rho_{Q_1}(X) \leq 0. \end{gathered} \tag{9}$$

Using a generalized duality approach we show in Theorem 3.2.3 that a solution to this problem is given by

$$X_{P,Q_1,Q_0} := x^* \left(\lambda_1^* \frac{dQ_1}{dQ}, \lambda_2^* \frac{dP}{dQ} \right), \tag{10}$$

where x^* is the solution to a deterministic maximization problem, and λ_1^* and λ_2^* are suitable real parameters that ensure that the constraints are satisfied. If the risk constraint is not binding, x^* reduces to the function I from (7). The most challenging part here is to show the existence of the Lagrange multipliers λ_1^* and λ_2^*, which implies the existence of the solution (10). This result is given in Lemma 2.3.4.

Then we consider the general problem (8) for utility functions that are only defined on the positive halfline. We solve a generalized projection problem which leads to three measures P^*, Q_1^*, and Q^*, that are characterized as worst case measures for Problem (8) in Proposition 3.3.12. This characterization then allows us to solve the robust utility maximization problem under both a budget and a risk constraint. The solution, which is given in Theorem 3.3.13, is of the form

$$X^* := x^* \left(\lambda_1^* \frac{dQ_1^*}{dQ^*}, \lambda_2^* \frac{dP^*}{dQ^*} \right). \tag{11}$$

In Section 3.4 we compare the optimal claim X^* with the optimal claims without a risk constraint, and under a value-at-risk constraint. We show that our risk constraint decreases the size and the probability of a loss, whereas the value-at-risk constraint only reduces the probability of a loss, but may entail very large losses that occur with a small probability.

Some of the results of Chapter 3 are also contained in a joint paper with Stefan Weber [2005].

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Chapter 1

Existence of Robust f-Projections in the Class of Martingale Measures

In this chapter we solve the problem of minimizing the f-divergence over two sets of measures. The solution will be used to treat the utility maximization problem in Chapter 2. Before we explain the problem, let us start by introducing f-divergences and f-projections.

Let $(\Omega, \mathcal{F})$ be a measurable space and denote by $\mathcal{M}_1(\Omega)$ the set of probability measures on $(\Omega, \mathcal{F})$. Let the function $f : [0, \infty) \to \mathbb{R} \cup \{\infty\}$ be convex and continuous. This means that possibly $f(0) = \infty$, but $f(x) < \infty$ for all $x > 0$. In order to define the f-divergence of $P \in \mathcal{M}_1(\Omega)$ with respect to $Q \in \mathcal{M}_1(\Omega)$, we associate to $f(\cdot)$ the function $f(\cdot, \cdot)$ on $[0, \infty) \times [0, \infty)$ defined by

$$f(x, y) := \begin{cases} 0 & \text{if } x = y = 0, \\ x \lim\limits_{z \to \infty} \frac{f(z)}{z} & \text{if } y = 0, x > 0, \\ y f\left(\frac{x}{y}\right) & \text{if } y > 0. \end{cases} \tag{1.1}$$

For an affine function $l(x) = ax+b$ on $[0, \infty)$, the associated function $l(\cdot, \cdot)$ on $[0, \infty) \times [0, \infty)$ is given by $l(x, y) = ax + by$. Since $f(\cdot, \cdot)$ is the supremum of the affine functions $l(\cdot, \cdot)$ associated to some affine function l on $[0, \infty)$ such that $l \leq f$, $f(\cdot, \cdot)$ is lower semicontinuous and convex on $[0, \infty) \times [0, \infty)$.

Definition 1.0.1. *Let P, $Q \in \mathcal{M}_1(\Omega)$, and let $R \in \mathcal{M}_1(\Omega)$ be some reference measure such that $P, Q \ll R$; for example, we may take $R := (P + Q)/2$. The* f-divergence *of P with respect to Q is defined as*

$$f(P|Q) := \int f\left(\frac{dP}{dR}, \frac{dQ}{dR}\right) dR = E_R\left[f\left(\frac{dP}{dR}, \frac{dQ}{dR}\right)\right],$$

where E_R denotes the expectation under the measure R.

Remark 1.0.2. *Let P^a and P^s denote the absolutely continuous and singular part in the Hahn-Lebesgue decomposition of $P \in \mathcal{M}_1(\Omega)$ with respect to $Q \in \mathcal{M}_1(\Omega)$. Then*

$$f(P|Q) = \int f\left(\frac{dP^a}{dQ}\right) dQ + \lim_{x\to\infty} \frac{f(x)}{x} \cdot P^s(\Omega) \quad \in (-\infty, \infty]; \tag{1.2}$$

note that the first term on the right-hand side is bounded from below by $f(P^a(\Omega))$ due to Jensen's inequality and that $\lim_{x\to\infty} f(x)/x > -\infty$. In particular the f-divergence is well defined, and it is independent of the choice of the reference measure R. If $P \ll Q$ or if $\lim_{x\to\infty} f(x)/x = 0$, then Equation (1.2) *reduces to*

$$f(P|Q) = \int f\left(\frac{dP^a}{dQ}\right) dQ \quad \in [f(P^a(\Omega)), \infty].$$

On the other hand, if $\lim_{x\to\infty} f(x)/x = \infty$, then $P \ll Q$ as soon as $f(P|Q) < \infty$.

Due to Jensen's inequality, the f-divergence may be interpreted as a measure of distance between probability distributions, and we call minimizing measures f-projections:

Definition 1.0.3. *For a subset $\mathcal{P}$ of $\mathcal{M}_1(\Omega)$ and a measure $Q \in \mathcal{M}_1(\Omega)$, $P_Q \in \mathcal{P}$ is called an f-*projection *of Q on $\mathcal{P}$ if it minimizes the f-divergence over the set $\mathcal{P}$:*

$$f(P_Q|Q) = f(\mathcal{P}|Q) := \inf_{P\in\mathcal{P}} f(P|Q).$$

For a subset $\mathcal{Q}$ of $\mathcal{M}_1(\Omega)$ and $P \in \mathcal{M}_1(\Omega)$, $Q_P \in \mathcal{Q}$ is called a reverse f-projection *of P on $\mathcal{Q}$ if it minimizes the f-divergence of P over the set $\mathcal{Q}$:*

$$f(P|Q_P) = f(P|\mathcal{Q}) := \inf_{Q\in\mathcal{Q}} f(P|Q).$$

Finally, $P^* \in \mathcal{P}$ is called a robust f-projection *of $\mathcal{Q}$ on $\mathcal{P}$ if it minimizes the* robust f-divergence *$f(P|\mathcal{Q}) := \inf_{Q\in\mathcal{Q}} f(P|Q)$ over the set $\mathcal{P}$:*

$$f(P^*|\mathcal{Q}) = f(\mathcal{P}|\mathcal{Q}) := \inf_{P\in\mathcal{P}} f(P|\mathcal{Q}),$$

i.e.,

$$\inf_{Q\in\mathcal{Q}} f(P^*|Q) = \inf_{P\in\mathcal{P}} \inf_{Q\in\mathcal{Q}} f(P|Q).$$

Example 1.0.4. *For $f(x) = x\log x$, the f-divergence equals the* relative entropy

$$H(P|Q) := \begin{cases} E_Q\left[\frac{dP}{dQ}\log\left(\frac{dP}{dQ}\right)\right] & \text{if } P \ll Q, \\ \infty & \text{otherwise.} \end{cases}$$

For $f(x) = -\log x$, we obtain the reverse relative entropy

$$f(P|Q) = H(Q|P) = \begin{cases} E_Q\left[\log\left(\frac{dQ}{dP}\right)\right] & \text{if } Q \ll P, \\ \infty & \text{otherwise.} \end{cases}$$

Other common examples include the power functions $f(x) = x^p$ for $p > 1$ or $p < 0$.

Remark 1.0.5. *Define the convex continuous function $\hat{f} : [0,\infty) \to \mathbb{R}\cup\{\infty\}$ by*

$$\hat{f}(x) := x \cdot f\left(\frac{1}{x}\right).$$

Then $f(P|Q) = \hat{f}(Q|P)$, and a reverse f-projection of P on $\mathcal{Q}$ may be viewed as an $\hat{f}$-projection of P on $\mathcal{Q}$; see Liese and Vajda [1987], Theorem 1.13. If f is strictly convex, then so is $\hat{f}$. In this case there is at most one f-projection P_Q of Q on $\mathcal{P}$ and at most one reverse f-projection Q_P of P on $\mathcal{Q}$.

Let us now fix two convex subsets $\mathcal{P}$ and $\mathcal{Q}$ of measures in $\mathcal{M}_1(\Omega)$ that are absolutely continuous with respect to some reference measure R. Our aim is to prove the existence of a robust f-projection P^* of $\mathcal{Q}$ on $\mathcal{P}$ and its reverse f-projections Q^* under suitable conditions on these sets. The definition (1.1) of the function $f(\cdot,\cdot)$ suggests that the limit

$$\frac{f(\infty)}{\infty} := \lim_{x\to\infty} \frac{f(x)}{x}$$

plays a crucial role in the analysis of this minimization problem, and this is indeed the case. In Section 1.1 we consider convex functions f with $f(\infty)/\infty = \infty$, and in Section 1.2 we solve the minimization problem for the case $f(\infty)/\infty = 0$. The methods we use in the two sections are completely different.

In Section 1.1 we use the property

$$\frac{f(\infty)}{\infty} = \infty \tag{1.3}$$

to construct an auxiliary convex function l also satisfying (1.3) such that $E_R[l(dP/dR)]$ is bounded whenever $f(P|Q)$ is bounded for some $Q \in \mathcal{Q}$.

The de la Vallée-Poussin criterion then implies that the subset $\{P \in \mathcal{P} : f(P|\mathcal{Q}) \leq c\}$ is weakly compact if $\mathcal{P}$ is closed. If $\mathcal{Q}$ is compact in a suitable sense, then one can use the lower semicontinuity of the f-divergence to prove the existence of P^* and Q^*. The most challenging parts are the construction of the function l and the proof of the boundedness of the expectations $E_R[l(dP/dR)]$. Here we use Young's inequality, which is a generalization of Hölder's inequality: Instead of L^p-spaces, one considers Orlizc spaces defined via suitable convex functions.

$\mathcal{P}$ may be interpreted as the set of absolutely continuous martingale measures for some semimartingale which models the stock price processes in a financial market, and this interpretation is used when the results are applied in Chapter 2. But we should keep in mind that they hold for general sets $\mathcal{P}$ that satisfy the assumptions below.

In Section 1.2, where we consider convex functions f satisfying

$$\frac{f(\infty)}{\infty} = 0, \tag{1.4}$$

we are more concrete about the probability space, and we let $\mathcal{P}$ be indeed the set of absolutely continuous martingale measures for some semimartingale. We could, of course, only assume that $\mathcal{P}$ is any set satisfying the required claims. But these claims are rather specific, and a large part of this section is in fact devoted to their proofs. In order to guarantee the existence of a robust f-projection, we have to enlarge the class $\mathcal{P}$ and consider the minimization problem over this enlarged class. To this end, we identify a martingale measure P with its density process and embed the space of density processes into a certain set of supermartingales. Föllmer [1972] showed that to any such supermartingale there is a probability measure on the product space $\Omega \times (0, \infty]$ equipped with the σ-field of predictable sets, and this gives us our extended class of martingale measures. We will show that a robust f-projection in this class exists if the set $\mathcal{Q}$ is compact in a suitable sense. To this end, we prove that the class of extended martingale measures is closed with respect to the almost sure convergence. Then the existence will follow from the uniform integrability of the negative parts of the variables random $f(dP/dQ)$.

Before we start, let us cite some results that will be used repeatedly in the following.

Theorem 1.0.6 (Dellacherie and Meyer [1975], Theorem II.22)**.** *A subset $\mathcal{K} \subseteq L^1(R)$ is uniformly integrable if and only if there is a function $g : [0, \infty) \to [0, \infty)$ with $\lim_{x\to\infty} g(x)/x = \infty$ such that*

$$\sup_{X \in \mathcal{K}} E_R[g(|X|)] < \infty.$$

This property is also called the de la Vallée-Poussin criterion. The following equivalence is also referred to as Dunford-Pettis compactness criterion.

Theorem 1.0.7 (Extract from Dellacherie and Meyer [1975], Theorem II.25)**.** *A subset $\mathcal{K} \subseteq L^1(R)$ is uniformly integrable if and only if it is relatively compact in the weak topology on $L^1(R)$.*

Remark 1.0.8. *Here and in the following, by the weak topology on $L^1(R)$ we mean the $\sigma(L^1(R), L^\infty(R))$-topology, which is the weakest topology such that all mappings*

$$\zeta \in L^1(R) \mapsto \int \zeta \eta dR, \qquad \eta \in L^\infty(R),$$

are continuous. We will sometimes speak of weakly compact sets of measures, and by this we mean that a corresponding set of densities is weakly compact in the above sense.

The following result is a special case of Theorem V.3.13 in Dunford and Schwartz [1958].

Theorem 1.0.9 (Dunford and Schwartz [1958])**.** *A convex subset of $L^1(R)$ is weakly closed if and only if it is strongly closed.*

1.1 The Existence Result for the Case $f(\infty)/\infty = \infty$

In this section we assume that $f : [0, \infty) \to \mathbb{R} \cup \{\infty\}$ is a convex continuous function with

$$\lim_{x\to\infty} \frac{f(x)}{x} = \infty. \tag{1.5}$$

Our aim is to show that a robust f-projection of $\mathcal{Q}$ on $\mathcal{P}$ exists under the following assumptions.

Assumption 1.1.1. *All measures in $\mathcal{P}$ and $\mathcal{Q}$ are absolutely continuous with respect to some reference measure R. The convex set*

$$\mathcal{K}_{\mathcal{P}} := \left\{ \frac{dP}{dR} : P \in \mathcal{P} \right\}$$

is closed in $L^1(R)$, and the convex set

$$\mathcal{K}_{\mathcal{Q}} := \left\{ \frac{dQ}{dR} : Q \in \mathcal{Q} \right\}$$

is weakly compact in $L^1(R)$.

Note that $\mathcal{K}_{\mathcal{P}}$ is closed in $L^1(R)$ if and only if $\mathcal{P}$ is closed in variation. Since the set $\mathcal{K}_{\mathcal{P}}$ is convex, it is closed in $L^1(R)$ if and only if it is weakly closed due to Theorem 1.0.9 by Dunford and Schwartz [1958].

Let us now state the main result of this section.

Theorem 1.1.2. *Let Assumption 1.1.1 hold and assume furthermore that the convex continuous function f satisfies* (1.5)*. Then there exists a robust f-projection P^* of $\mathcal{Q}$ on $\mathcal{P}$. Moreover, there exists a reverse f-projection Q^* of P^* on $\mathcal{Q}$, i.e.,*

$$f(P^*|Q^*) = f(P^*|\mathcal{Q}) = f(\mathcal{P}|\mathcal{Q}).$$

The proof consists of three steps: First we show that the f-divergence is jointly lower semicontinuous in P and Q, then we formulate a compactness criterion in terms of some auxiliary function l, and in the third step we construct such a function l which has the required properties.

Define

$$F_R(\phi, \psi) := \int f(\phi, \psi) dR \tag{1.6}$$

for $\mathcal{F}$-measurable $\phi, \psi \geq 0$. Note that $f(\phi, \psi) \geq b\psi$ for some constant b since $f(\cdot)$ is convex and finally increasing due to our assumption (1.5), hence bounded from below on $[0, \infty)$. Thus $F_R(\phi, \psi) \in (-\infty, \infty]$ is well defined. Note also that

$$f(P|Q) = F_R\left(\frac{dP}{dR}, \frac{dQ}{dR}\right)$$

for P, Q, $R \in \mathcal{M}_1(\Omega)$ such that P, $Q \ll R$. We will view F_R as a functional on the closed convex subset $L^1_+(R) \times L^1_+(R)$ of the Banach space $L^1(R) \times L^1(R)$.

The following result appears also in Liese and Vajda [1987], Theorem 1.47, but with a different proof.

Lemma 1.1.3. *Under Assumption* (1.5) *the functional F_R is convex and weakly lower semicontinuous on $L^1_+(R) \times L^1_+(R)$.*

Proof. Convexity of F_R follows from the convexity of $f(\cdot, \cdot)$ on $[0, \infty)^2$. In order to verify weak lower semicontinuity, we have to show that the sets

$$A_c := \big\{(\phi, \psi) \in L^1_+(R) \times L^1_+(R) : F_R(\phi, \psi) \leq c\big\}$$

are closed with respect to the weak product topology on $L^1(R) \times L^1(R)$. But since A_c is convex, it is enough to check that A_c is strongly closed due to Theorem 1.0.9 by Dunford and Schwartz [1958]. To this end, take $(\phi_n, \psi_n) \in A_c$ $(n \geq 1)$ such that $\phi_n \to \phi$ and $\psi_n \to \psi$ in $L^1(R)$ as n tends to infinity. Passing to subsequences if necessary, we may assume that both sequences converge R-almost surely. Since $f(\phi_n, \psi_n) \geq b\psi_n$ for some $b \in \mathbb{R}$ and $(\psi_n)_{n=1,2,\ldots}$ is uniformly integrable, we can use the lower semicontinuity of f on $[0,\infty)^2$ and Fatou's lemma to conclude

$$\begin{aligned} F_R(\phi, \psi) &= \int f(\lim_{n\to\infty}(\phi_n, \psi_n))dR \\ &\leq \int \liminf_{n\to\infty} f(\phi_n, \psi_n)dR \\ &\leq \liminf_{n\to\infty} F_R(\phi_n, \psi_n) \leq c. \end{aligned}$$

Hence we have $(\phi, \psi) \in A_c$. □

Remark 1.1.4. *In particular the functional $F_R(dP/dR, \cdot)$ is weakly lower semicontinuous on the weakly compact set $\mathcal{K}_\mathcal{Q}$. This shows that a reverse f-projection Q_P of P on $\mathcal{Q}$ exists for any $P \in \mathcal{M}_1(\Omega)$. Thus the existence of a robust f-projection of $\mathcal{Q}$ on $\mathcal{P}$ amounts to the existence of some $P^* \in \mathcal{P}$ which minimizes the f-divergence $f(P|Q_P)$ over $\mathcal{P}$.*

Since $F_R(\cdot,\cdot)$ is weakly lower semicontinuous on $\mathcal{K}_\mathcal{P} \times \mathcal{K}_\mathcal{Q}$, the existence of a robust f-projection will now follow if we can show that the set $\{(P,Q) : f(P|Q) \leq c\}$ is compact in the weak product topology. To this end, we prove the following criterion.

Lemma 1.1.5. *Let $l : [0,\infty) \to \mathbb{R}$ be a positive increasing function with $\lim_{x\to\infty} l(x)/x = \infty$. Let Assumption 1.1.1 hold and assume that for any constant $c > 0$ there is a constant $c_0 > 0$ such that for any $P \in \mathcal{P}$*

$$f(P|\mathcal{Q}) \leq c \qquad \Longrightarrow \qquad E_R\left[l\left(\frac{dP}{dR}\right)\right] \leq c_0. \tag{1.7}$$

Then there exist a robust f-projection P^* of $\mathcal{Q}$ on $\mathcal{P}$ and a reverse f-projection Q^* of P^* on $\mathcal{Q}$.

Proof. We may assume $f(\mathcal{P}|\mathcal{Q}) < \infty$ because otherwise every $P \in \mathcal{P}$ would be a robust f-projection. Take $c > f(\mathcal{P}|\mathcal{Q})$. Since we have $f(P|Q) = F_R(dP/dR, dQ/dR)$ and since F_R is weakly lower semicontinuous by Lemma 1.1.3, it is enough to show that $\{(P,Q) \in \mathcal{P} \times \mathcal{Q} : f(P|Q) \leq c\}$, viewed as the subset

$$\mathcal{C}_c := \{(\phi, \psi) : F_R(\phi, \psi) \leq c\} \cap (\mathcal{K}_\mathcal{P} \times \mathcal{K}_\mathcal{Q})$$

of $L^1(R) \times L^1(R)$, is weakly compact. Then F_R attains its minimum in some $(P^*, Q^*) \in \mathcal{P} \times \mathcal{Q}$, which implies

$$f(P^*|\mathcal{Q}) = f(P^*|Q^*) = \inf_{P\in\mathcal{P}} f(P|\mathcal{Q}),$$

and so P^* is a robust f-projection of $\mathcal{Q}$ on $\mathcal{P}$, and Q^* is its reverse f-projection.

Under Condition (1.7)

$$\mathcal{C}_c \subseteq \mathcal{K}_{\mathcal{P},c_0} \times \mathcal{K}_{\mathcal{Q}},$$

where

$$\mathcal{K}_{\mathcal{P},c_0} := \{\phi \in \mathcal{K}_{\mathcal{P}} : E_R[l(\phi)] \leq c_0\}$$

is uniformly integrable and hence relatively compact in the weak topology on $L^1(R)$ due to Theorems 1.0.6 and 1.0.7 by Dellacherie and Meyer. Since $\mathcal{K}_{\mathcal{Q}}$ is weakly compact by Assumption 1.1.1, Tychonov's theorem implies that $\mathcal{K}_{\mathcal{P},c_0} \times \mathcal{K}_{\mathcal{Q}}$ is relatively compact in the weak product topology, and so is $\mathcal{C}_c$. But $\mathcal{C}_c$ is also weakly closed due to the lower semicontinuity of F_R and Assumption 1.1.1, and so $\mathcal{C}_c$ is in fact weakly compact. □

Remark 1.1.6. *Consider the classical case $\mathcal{Q} = \{Q_0\}$. Then Condition* (1.7) *is trivially satisfied for $l = f$ and $R = Q_0$, and the preceding proof reduces to the standard argument for the existence of a classical f-projection; see, e.g., Liese and Vajda [1987], Proposition 8.5, and in the relative entropy case $f(x) = x \log x$ also Csiszár [1975], Theorem 2.1.*

In our main proof we will need a generalization of the Hölder inequalities. To this end, we introduce Young functions and Young's inequality, which can be found in the Appendix of Neveu [1972].

Definition 1.1.7. *A* Young function *is a function $h : [0,\infty) \to [0,\infty)$ that is continuous, increasing, convex and zero at the origin. The* conjugate Young function *$h^* : [0,\infty) \to [0,\infty)$ is defined by the Fenchel-Legendre transform*

$$h^*(x) := \sup_{y\geq 0}\{xy - h(y)\}. \tag{1.8}$$

We have $h(x) = \sup_{y\geq 0}\{xy - h^*(y)\}$, and it is straightforward to check that h^* has the same properties as h (see also Neveu [1972], pages 193 and 194). (h, h^*) is called a Young couple.

For a probability measure $Q \in \mathcal{M}_1(\Omega)$, we define the space

$$L^h(Q) = L^h(\Omega, \mathcal{F}, Q) := \left\{X \text{ on } (\Omega, \mathcal{F}) : \exists\, a > 0 : E_Q\left[h\left(\frac{|X|}{a}\right)\right] \leq 1\right\}. \tag{1.9}$$

Note that in the special case $h(x) = x^p$ we have $L^h = L^p$.

Proposition 1.1.8 (Extracts from Neveu [1972], Proposition IX.2.2)**.** *Let (h, h^*) be a Young couple and let $Q \in \mathcal{M}_1(\Omega)$.*

(i) *We have $L^\infty(Q) \subseteq L^h(Q) \subseteq L^1(Q)$, and*

$$||X||_{Q,h} := \inf\left\{a > 0 : E_Q\left[h\left(\frac{|X|}{a}\right)\right] \leq 1\right\} \tag{1.10}$$

defines a norm on $L^h(Q)$.

(ii) *The space $L^h(Q)$ is complete, and we call this Banach space an* Orlicz space.

(iii) (Young's inequality) *For $X \in L^h(Q)$ and $Y \in L^{h^*}(Q)$, we have*

$$E_Q[|XY|] \leq 2 \cdot ||X||_{Q,h} \cdot ||Y||_{Q,h^*}. \tag{1.11}$$

(iv) *Furthermore, we have*

$$||X||_{Q,h} \leq \max\{1, E_Q[h(|X|)]\} \tag{1.12}$$

for $X \in L^h(Q)$ (this statement only appears in Neveu's proof of this proposition).

Since $\mathcal{K}_{\mathcal{Q}}$ is assumed to be weakly compact, we can choose a function $g : [0, \infty) \to [0, \infty)$ with $\lim_{x\to\infty} g(x)/x = \infty$ such that

$$\sup_{Q\in\mathcal{Q}} E_R\left[g\left(\frac{dQ}{dR}\right)\right] < \infty, \tag{1.13}$$

due to Theorem 1.0.6 by Dellacherie and Meyer [1975]. Given the functions f and g, we are now going to construct a suitable function l and at the same time a convex function h such that an appropriate Young inequality with respect to h will allow us to obtain the estimate in terms of l which is required in Lemma 1.1.5.

Lemma 1.1.9. *There exist strictly increasing functions h and l_i $(i = 1, 2)$ on $[0, \infty)$ with initial value $h(0) = l_i(0) = 0$ such that the following properties hold:*

(i) *h is continuous, convex, strictly increasing, and $\lim_{x\to\infty} h(x)/x = \infty$.*

(ii) *l_i is concave and $\lim_{x\to\infty} l_i(x) = \infty$ $(i = 1, 2)$.*

(iii) *$h(xl_1(x)) \leq f(x)$ for large enough x.*

(iv) *$xh^*(l_2(x)) \leq g(x)$ for large enough x.*

(v) $l(x) := xl_1(l_2(x)) \leq g(x)$ *for large enough* x.

Proof. We are going to use repeatedly the following simple fact: If $\tilde{u}$ is a function on $[0,\infty)$ such that $\lim_{x\to\infty}\tilde{u}(x)=\infty$, then there is a strictly increasing concave function u on $[0,\infty)$ such that $\lim_{x\to\infty}u(x)=\infty$, $u(0)=0$, and $u(x) \leq \tilde{u}(x)$ on $[x_1,\infty)$ for some $x_1 \geq 0$. Indeed, take a sequence $0 = x_0 \leq x_1 < x_2 < \dots$ converging to infinity such that for $n \geq 1$, $\tilde{u}(x) \geq n+1$ for all $x \geq x_n$, and the sequence $x_{n+1}-x_n$ increases in $n \geq 0$. Define $u(x_n) := n$ and u linear between x_n and x_{n+1} for $n \geq 0$. Then we have $u(x) \leq n+1 \leq \tilde{u}(x)$ on $[x_n, x_{n+1})$ for $n \geq 1$, hence u is dominated by $\tilde{u}$ on $[x_1,\infty)$. Furthermore, $u'(x) = (u(x_{n+1})-u(x_n))/(x_{n+1}-x_n) = 1/(x_{n+1}-x_n)$ for $x \in (x_n, x_{n+1})$ for $n \geq 0$. Since this fraction is non-increasing, u is concave.

In a first step we construct the convex function h. Since the function f is convex and $\lim_{x\to\infty} f(x)/x = \infty$, its left-hand derivative f'_- is non-decreasing and tends to infinity. In particular $f'_- > 0$ on $[x_0,\infty)$ for some $x_0 \geq 0$. Take a non-decreasing function $\zeta : [0,\infty) \to [0,\infty)$ that tends to infinity, but satisfies $\lim_{x\to\infty}\zeta(x)/x = 0$. Define

$$h'(x) := \gamma(x) f'_-(\zeta(x)) \tag{1.14}$$

on $[x_0,\infty)$, where $\gamma : [0,\infty) \to [0,\infty)$ is decreasing, tending to 0, and such that $h' > 0$ is non-decreasing and tends to infinity. For example, we may choose $\zeta(x) := \sqrt{x}$ and $\gamma(x) := (f'_-(\zeta(x)))^{-1/2}$.

Now define h such that (1.14) is satisfied on $[x_0,\infty)$, and h is linear on $[0,x_0)$ with $h(0)=0$ and $h(x_0) = x_0 h'(x_0)$. Then h is a convex function which has the required properties. Moreover,

$$\lim_{x\to\infty}\frac{h(cx)}{f(x)} = 0 \qquad \text{for all } c > 0. \tag{1.15}$$

Indeed, for $c \in (0,\infty)$ take $\alpha \geq x_0$ such that $\zeta(y) \leq y/c$ for $y \geq \alpha$. Then we have for $cx \geq \alpha$,

$$\begin{aligned} h(cx) &= h(\alpha) + \int_\alpha^{cx} \gamma(y) f'_-(\zeta(y))dy \\ &\leq h(\alpha) + \gamma(\alpha)\int_\alpha^{cx} f'_-\left(\frac{y}{c}\right)dy \\ &= h(\alpha) + \gamma(\alpha)c\left(f(x) - f\left(\frac{\alpha}{c}\right)\right). \end{aligned}$$

Therefore,

$$\limsup_{x\to\infty}\frac{h(cx)}{f(x)} \leq c\gamma(\alpha),$$

and this implies (1.15) since $\lim_{\alpha\to\infty}\gamma(\alpha)=0$.

In order to construct the concave function l_1, consider first the function $\tilde{l}_1$ defined by $h(x\tilde{l}_1(x))=f(x)$, i.e., $\tilde{l}_1(x):=h^{-1}(f(x))/x$. Then $\lim_{x\to\infty}\tilde{l}_1(x)=\infty$, because otherwise there would be a $c\in(0,\infty)$ and a sequence (x_n) tending to infinity such that

$$h(x_n c)\geq h(x_n\tilde{l}_1(x_n))=f(x_n),$$

in contradiction to (1.15). As explained above, we can now choose a strictly increasing concave function l_1 such that $l_1(0)=0$, $\lim_{x\to\infty}l_1(x)=\infty$, and $l_1(x)\leq\tilde{l}_1(x)$, hence $h(xl_1(x))\leq f(x)$ for large enough x.

Finally, we construct the concave function l_2. Let h^* be the Fenchel-Legendre transform of h defined in (1.8), which has the the same properties as h specified in (i). First we define $\tilde{l}_2(x)$ on $[0,\infty)$ such that

$$h^*(\tilde{l}_2(x))=\frac{g(x)}{x}\quad\text{ on }(0,\infty).$$

This implies $\lim_{x\to\infty}\tilde{l}_2(x)=\infty$. We can now choose a strictly increasing concave function l_2 such that $l_2(0)=0$, $\lim_{x\to\infty}l_2(x)=\infty$ and $l_2(x)\leq\tilde{l}_2(x)\wedge l_1^{-1}(g(x)/x)$, hence $xh^*(l_2(x))\leq g(x)$ and $xl_1(l_2(x))\leq g(x)$, for large enough x. $\square$

In order to conclude the proof of Theorem 1.1.2, we now show that the function l appearing in part (v) of Lemma 1.1.9 allows us to apply the criterion in Lemma 1.1.5.

Lemma 1.1.10. *The function l defined in Lemma 1.1.9 satisfies the conditions of Lemma 1.1.5.*

Proof. Observe first that $\lim_{x\to\infty}l(x)/x=\infty$. Now let us fix $P\in\mathcal{P}$, and $Q\in\mathcal{Q}$ such that $f(P|Q)\leq c$ for some $c>0$. Then $P\ll Q$ by Remark 1.0.2, and $\phi:=dP/dQ$ and $\psi:=dQ/dR$ are well defined. Let $x_0>1$ be such that Conditions (iii)-(v) in Lemma 1.1.9 are satisfied for $x\geq x_0$. In order to verify Condition (1.7) we decompose the expectation on the right-hand side as follows:

$$\begin{aligned}E_R\left[l\left(\frac{dP}{dR}\right)\right]&=E_R[l(\phi\psi)]\\&=E_R[l(\phi\psi);\phi\leq x_0]+E_R[l(\phi\psi);\phi>x_0,l_2(\psi)>1]\\&\quad+E_R[l(\phi\psi);\phi>x_0,l_2(\psi)\leq 1].\end{aligned}\tag{1.16}$$

We are going to show that each of these three terms is bounded by some constant which only depends on c but not on the specific choice of P and Q.

Since l_i is concave with $l_i(0) = 0$ for $i = 1, 2$, we have $l_i(\alpha x) \leq \alpha l_i(x)$ for any $\alpha \geq 1$, and this estimate will be used repeatedly.

On $\{\phi \leq x_0\}$ we have

$$\begin{aligned} l(\phi\psi) &\leq l(x_0\psi) \\ &= x_0\psi l_1(l_2(x_0\psi)) \\ &\leq x_0^2\psi l_1(l_2(\psi)) \\ &= x_0^2 l(\psi) \leq x_0^2(c_1 + g(\psi)), \end{aligned}$$

where $c_1 := \sup\{l(x) : x \leq x_0\}$, since $l(x) \leq g(x)$ for $x \geq x_0$. So the first term above satisfies

$$E_R[l(\phi\psi); \phi \leq x_0] \leq x_0^2\,(c_1 + E_R[g(\psi)]) \leq x_0^2 \left(c_1 + \sup_{Q\in\mathcal{Q}} E_R\left[g\left(\frac{dQ}{dR}\right)\right]\right),$$

which is finite by (1.13).

On $\{\phi > x_0,\ l_2(\psi) > 1\}$ we have

$$l_1(l_2(\phi\psi)) \leq l_1(\phi l_2(\psi)) \leq l_1(\phi)l_2(\psi),$$

and this implies

$$E_R[l(\phi\psi); \phi > x_0, l_2(\psi) > 1] \leq E_Q[\phi l_1(\phi)l_2(\psi)].$$

Now we use Young's inequality (1.11) to conclude that

$$E_Q[\phi l_1(\phi)l_2(\psi)] \leq 2 \cdot ||\phi l_1(\phi)||_{Q,h} \cdot ||l_2(\psi)||_{Q,h^*}.$$

But

$$||\phi l_1(\phi)||_{Q,h} \leq \max\{1, E_Q[h(\phi l_1(\phi))]\}$$

by (1.12), and

$$\begin{aligned} E_Q[h(\phi l_1(\phi))] &\leq c_2 + E_Q[f(\phi)] \\ &= c_2 + f(P|Q) \\ &\leq c_2 + c, \end{aligned}$$

where $c_2 := \sup\{h(xl_1(x)) : x \leq x_0\}$, since $h(xl_1(x)) \leq f(x)$ for $x \geq x_0$. In the same way,

$$||l_2(\psi)||_{Q,h^*} \leq \max\{1, E_Q[h^*(l_2(\psi))]\},$$

and

$$\begin{aligned} E_Q[h^*(l_2(\psi))] &= E_R[\psi h^*(l_2(\psi))] \\ &\leq c_3 + E_R[g(\psi)] \\ &\leq c_3 + \sup_{Q\in\mathcal{Q}} E_R\left[g\left(\frac{dQ}{dR}\right)\right], \end{aligned}$$

where $c_3 := \sup\{xh^*(l_2(x)) : x \leq x_0\}$, since $xh^*(l_2(x)) \leq g(x)$ for $x \geq x_0$. This yields the desired bound for the second term on the right-hand side of Equation (1.16).

On $\{\phi > x_0,\ l_2(\psi) \leq 1\}$ we have

$$l_1(l_2(\phi\psi)) \leq l_1(\phi l_2(\psi)) \leq l_1(\phi),$$

and so the remaining term satisfies

$$E_R[l(\phi\psi); \phi > x_0, l_2(\psi) < 1] \leq E_R[\phi\psi l_1(\phi)] = E_Q[\phi l_1(\phi)].$$

Young's inequality yields

$$E_Q[\phi l_1(\phi)] \leq 2 \cdot ||\phi l_1(\phi)||_{Q,h} \cdot \inf\left\{a > 0 : h^*\left(\frac{1}{a}\right) \leq 1\right\},$$

and we have already seen above that $||\phi l_1(\phi)||_{Q,h}$ is suitably bounded. □

Remark 1.1.11. *For special choices of functions f and g, the construction of our auxiliary function l may of course be simpler. Take for example $f(x) = x^\alpha$ and $g(x) = x^\beta$ with $\alpha,\ \beta > 1$. Choose $\gamma > 1$ such that $\gamma < \alpha$ and $(\alpha-1)\gamma \leq \beta(\alpha-\gamma)$ and define $l(x) = x^\gamma$. Condition* (1.7) *now follows by applying Hölder's inequality with exponents $p = \alpha/\gamma$ and $q = \alpha/(\alpha-\gamma)$: For $P \in \mathcal{P}$, $Q \in \mathcal{Q}$, and $\phi := dP/dQ$, $\psi := dQ/dR$, we have*

$$\begin{aligned} E_R\left[l\left(\frac{dP}{dR}\right)\right] &= E_R\left[\phi^\gamma\psi^\gamma\right] = E_Q\left[\phi^\gamma\psi^{\gamma-1}\right] \\ &\leq E_Q\left[\phi^{\gamma p}\right]^{1/p} E_Q\left[\psi^{(\gamma-1)q}\right]^{1/q} \\ &\leq f(P|Q)^{1/p}\left(1 + E_R\left[g\left(\frac{dQ}{dR}\right)\right]^{1/q}\right). \end{aligned}$$

Now the proof of Theorem 1.1.2 can easily be completed:

Proof of Theorem 1.1.2. Due to Lemma 1.1.10, we can apply Lemma 1.1.5 to conclude that a robust f-projection P^* of $\mathcal{Q}$ on $\mathcal{P}$ and a reverse f-projection Q^* of P^* on $\mathcal{Q}$ exist. □

We conclude this section with a uniqueness result for robust f-projections. In order to define the density of $P \in \mathcal{P}$ with respect to $Q \in \mathcal{Q}$ also R-almost surely, we set

$$\frac{dP}{dQ} := \frac{dP}{dR}\Big/\frac{dQ}{dR}1_{\{dQ/dR>0\}} + \infty \cdot 1_{\{dQ/dR=0, dP/dR>0\}}.$$

Proposition 1.1.12. *If f is strictly convex and $f(\mathcal{P}|\mathcal{Q}) < \infty$, then the density of the robust f-projection P^* of $\mathcal{Q}$ on $\mathcal{P}$ with respect to its reverse f-projection Q^* is R-almost surely unique.*

Proof. Assume that P_1 and $P_2 \in \mathcal{P}$ are two robust f-projections of $\mathcal{Q}$ on $\mathcal{P}$ with reverse f-projections Q_1 and Q_2. Then $P_i \ll Q_i$ due to Remark 1.0.2. Take $\gamma \in (0,1)$ and define $P_\gamma := \gamma P_1 + (1-\gamma)P_2$, $Q_\gamma := \gamma Q_1 + (1-\gamma)Q_2$, $\phi_i := dP_i/dQ_i$, and $\psi_i := dQ_i/dQ_\gamma$ for $i = 1, 2$. Note that $\gamma\psi_1 + (1-\gamma)\psi_2 = 1$ and $\gamma\psi_1\phi_1 + (1-\gamma)\psi_2\phi_2 = dP_\gamma/dQ_\gamma$. By convexity of f and minimality of P_1 and P_2,

$$\begin{aligned} f(P_\gamma|\mathcal{Q}) &\geq \gamma f(P_1|\mathcal{Q}) + (1-\gamma) f(P_2|\mathcal{Q}) \\ &= E_{Q_\gamma}\left[\gamma\psi_1 f(\phi_1) + (1-\gamma)\psi_2 f(\phi_2)\right] \\ &\geq E_{Q_\gamma}\left[f\left(\gamma\psi_1\phi_1 + (1-\gamma)\psi_2\phi_2\right)\right] \\ &= f(P_\gamma|Q_\gamma) \\ &\geq f(P_\gamma|\mathcal{Q}), \end{aligned}$$

and so we have equality everywhere. But since f is strictly convex, the second inequality can only reduce to an equality if $\phi_1 = \phi_2$ Q_γ-almost surely. This means that $\phi_1 = \phi_2$ R-almost surely on the set $\{dQ_\gamma/dR > 0\}$. On the set $\{dQ_\gamma/dR = 0\}$ we have $dP_i/dR = 0$ for $i = 1, 2$ R-almost surely since $f(P_i|Q_i) < \infty$, hence $\phi_1 = \phi_2 = 0$ R-almost surely. □

1.2 The Existence Result for the Case $f(\infty)/\infty = 0$

In this section we assume that $f : [0, \infty) \to \mathbb{R} \cup \{\infty\}$ is a convex continuous function with

$$\lim_{x\to\infty} \frac{f(x)}{x} = 0. \tag{1.17}$$

We show that a robust f-projection exists in an enlarged class of martingale measures. As in the previous section, we need

Assumption 1.2.1. *All measures in $\mathcal{Q}$ are absolutely continuous with respect to some reference measure R. The convex set*

$$\mathcal{K}_\mathcal{Q} = \left\{ \frac{dQ}{dR} : Q \in \mathcal{Q} \right\}$$

is weakly compact in $L^1(R)$.

Here we do not need to assume that the set $K_{\mathcal{P}}$ is closed as in the previous section.

The aim is to apply this existence result to solve a robust utility maximization problem. To this end, we want to be more precise about our probability space. We fix a reference measures R such that Assumption 1.2.1 holds. Let us introduce a filtration $(\mathcal{F}_t)_{0\leq t\leq T}$ for some finite time horizon T. We assume that the filtration is right-continuous, that $\mathcal{F}_T = \mathcal{F}$, and that $\mathcal{F}_0$ is trivial for R and contains all sets with R-measure zero. Furthermore, let $S = (S_t)_{0\leq t\leq T}$ be an $\mathbb{R}^d$-valued semimartingale on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, R)$ which will later be interpreted as the price process of d risky assets in a financial market. Here we let $\mathcal{P}$ be the set of absolutely continuous martingale measures.

Definition 1.2.2. *A probability measure $P \ll R$ is called an* absolutely continuous martingale measure *if the semimartingale $(S_t)_{0\leq t\leq T}$ is a local martingale under P. If in addition $P \sim R$, then P is called an* equivalent martingale measure. *The class of absolutely continuous martingale measures will be denoted by $\mathcal{P}$, the class of equivalent martingale measures by $\mathcal{P}_e$.*

Let us assume that the set of absolutely continuous martingale measures is non-empty, i.e.,

$$\mathcal{P} \neq \emptyset.$$

For $x_0 > 0$, let us introduce the set $\mathcal{V}(x_0)$ of stochastic integrals with respect to S, which will later be interpreted as self-financing portfolios with initial value x_0. Let $\xi = (\xi_t)_{0\leq t\leq T}$ be a d-dimensional predictable, S-integrable process and define

$$V_t := x_0 + \int_0^t \xi_s dS_s \qquad (0 \leq t \leq T). \tag{1.18}$$

The family $\mathcal{V}(x_0)$ denotes all such non-negative stochastic integrals $V = (V_t)_{0\leq t\leq T}$ with initial value x_0.

A first idea might consist in minimizing the robust f-divergence over the set $\mathcal{P}$ of absolutely continuous martingale measures. However, the minimizing measure is in general not contained in the set $\mathcal{P}$; see the counterexample 5.1 by Kramkov and Schachermayer [1999] for the classical case $\mathcal{Q} = \{Q_0\}$. Instead, we will consider the problem of minimizing the robust f-divergence over some generalized class of martingale measures. To this end, we enlarge our initial probability space by introducing an additional *default time* ζ, defined as the second coordinate $\zeta(\omega, s) := s$ on the product space $\bar{\Omega} := \Omega \times (0, \infty]$. Define $\mathcal{F}_t := \mathcal{F}_T$ for $t > T$ and let

$$\bar{\mathcal{F}} := \sigma(\{A \times (t, \infty] : A \in \mathcal{F}_t, t \geq 0\})$$

denote the predictable σ-field on $\bar{\Omega}$; the predictable filtration $(\bar{\mathcal{F}}_t)_{t\geq 0}$ is defined in the same manner, i.e.,

$$\bar{\mathcal{F}}_t := \sigma(\{A \times (s, \infty] : A \in \mathcal{F}_s, 0 \leq s \leq t\}).$$

An adapted process $Y = (Y_t)_{t\geq 0}$ on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ will be identified with the adapted process $\bar{Y} = (\bar{Y}_t)_{t\geq 0}$ on $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{t\geq 0})$ defined by $\bar{Y}_t := Y_t 1_{\{\zeta > t\}}$, i.e.,

$$\bar{Y}_t(\omega, s) := Y_t(\omega) 1_{(t,\infty]}(s) \qquad (t \geq 0).$$

To a probability measure Q on $(\Omega, \mathcal{F})$ corresponds the probability measure $\bar{Q} := Q \times \delta_\infty$ on $(\bar{\Omega}, \bar{\mathcal{F}})$. Conversely, for any probability measure $\bar{Q}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ we define its projections Q^t on $(\Omega, \mathcal{F}_t)$ by

$$Q^t(A) := \bar{Q}(A \times (t, \infty]) \qquad (A \in \mathcal{F}_t,\ t \geq 0).$$

Note that Q^t is a measure on $(\Omega, \mathcal{F}_t)$, but not necessarily a probability measure since we might have $Q^t(\Omega) < 1$.

In order to introduce the class $\bar{\mathcal{P}}$ of extended martingale measures, let us denote by $\bar{\mathcal{V}}(x_0)$ the class of corresponding value processes $\bar{V} = (\bar{V}_t)_{t\geq 0}$ with $\bar{V}_t = V_t 1_{\{\zeta > t\}}$ $(t \geq 0)$ for some $V \in \mathcal{V}(x_0)$.

Definition 1.2.3. *A probability measure $\bar{P}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ will be called an* extended martingale measure *if*

(i) $P^t \ll R$ on $\mathcal{F}_t$ $(t \geq 0)$,

(ii) Under $\bar{P}$, any $\bar{V} \in \bar{\mathcal{V}}(x_0)$ is a supermartingale with respect to $(\bar{\mathcal{F}}_t)_{t\geq 0}$.

We denote by $\bar{\mathcal{P}}$ the class of all extended martingale measures and by $\mathcal{P}^T$ the class of projections P^T on $(\Omega, \mathcal{F}_T)$ of measures in $\bar{\mathcal{P}}$.

Clearly, for any martingale measure $P \in \mathcal{P}$ the corresponding measure $\bar{P} := P \times \delta_\infty$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ belongs to $\bar{\mathcal{P}}$.

We are going to use the representation of a right-continuous non-negative supermartingale $Z = (Z_t)_{t\geq 0}$ with $Z_0 = 1$ as a probability measure $\bar{P}^Z$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ such that

$$\bar{P}^Z(A \times (t, \infty]) = E_R[Z_t; A] \tag{1.19}$$

for $A \in \mathcal{F}_t$ and $t \geq 0$; see Föllmer [1972] and Föllmer [1973]. This requires a regularity assumption on the underlying filtration, for instance in the following form.

Assumption 1.2.4. $(\mathcal{F}_t)_{t\geq 0}$ *is the right-continuous modification of a standard system* $(\mathcal{F}_t^0)_{t\geq 0}$ *in the sense of Föllmer [1972], Appendix, i.e.,* (i) *each* $(\Omega, \mathcal{F}_t^0)$ *is a standard Borel space, and* (ii) *any decreasing sequence of atoms* A_i *of* $\mathcal{F}_{t_i}$ *for* $0 \leq t_1 \leq t_2 \leq ...$ *has a non-void intersection; see also Chapter V in Parthasarathy [1967].*

Remark 1.2.5. $(\Omega, \mathcal{G})$ *is a* standard Borel space *if there exists a Polish space* Ω_0 *equipped with the Borel* σ*-field* $\mathcal{G}_0$ *and a bijective measurable mapping* $g : \Omega \to \Omega_0$ *such that* g *preserves countable set operations.*

Remark 1.2.6. (i) *For any probability measure* $\bar{P}$ *on* $(\bar{\Omega}, \bar{\mathcal{F}})$ *whose projections satisfy Condition* (i) *of Definition 1.2.3, the adapted process* $Z = (Z_t)_{t\geq 0}$ *defined by*

$$Z_t := \frac{dP^t}{dR} \qquad (t \geq 0) \tag{1.20}$$

is a right-continuous non-negative supermartingale on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, R)$ *with* $Z_0 = 1$*. Conversely, by Theorem 1.5 in Föllmer [1972] any such supermartingale induces a unique probability measure* $\bar{P}^Z$ *on* $(\bar{\Omega}, \bar{\mathcal{F}})$ *via* (1.19) *if the underlying filtered space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0})$ *is rich enough, for example in the sense of Assumption 1.2.4; see also Föllmer [1973], Meyer [1972], Azéma and Jeulin [1976], and Stricker [1972]. For any supermartingale* $Y = (Y_t)_{t\geq 0}$ *on* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, R)$*, the process* $\bar{U} = (\bar{U}_t)_{t\geq 0}$ *defined by*

$$\bar{U}_t(\omega, s) := \frac{Y_t}{Z_t} 1_{\{Z_t \neq 0\}} 1_{(t,\infty]}(s)$$

is a $\bar{P}^Z$*-supermartingale. Conversely, if the process* $\bar{U}$ *with* $\bar{U}_t = U_t 1_{\{\zeta > t\}}$ *is a supermartingale under* $\bar{P}^Z$ *, then* $Y := UZ$ *is an* R*-supermartingale; see Föllmer [1972], Proposition 4.2.*

(ii) *Let* $\bar{P} = \bar{P}^Z$ *be a probability measure on* $(\bar{\Omega}, \bar{F})$ *such that* (1.19) *holds. It follows from part* (i) *that* $\bar{P}$ *is an extended martingale measure if and only if*

$$ZV \text{ is an } R\text{-supermartingale for any } V \in \mathcal{V}(x_0). \tag{1.21}$$

Thus our class $\bar{\mathcal{P}}$ *of extended martingale measures corresponds exactly to the class of supermartingales which appears in the duality approach of Kramkov and Schachermayer to the problem of maximizing expected utility in incomplete financial markets; see Kramkov and Schachermayer [1999], page 6.*

The following result is a consequence of Lemma 5.2 by Föllmer and Kramkov [1997].

Lemma 1.2.7. *Let $(\bar{P}_n)_{n\geq 1}$ be a sequence in the set $\bar{\mathcal{P}}$. Then there is a sequence $\bar{P}_{n,0} \in conv(\bar{P}_n, \bar{P}_{n+1}, ...)$ $(n = 1, 2, ...)$ and a measure $\bar{P}_0 \in \bar{\mathcal{P}}$ such that*

$$\left.\frac{dP_{n,0}^T}{dR}\right|_{\mathcal{F}_T} \longrightarrow \left.\frac{dP_0^T}{dR}\right|_{\mathcal{F}_T} \qquad R-\textit{almost surely.} \tag{1.22}$$

Proof. Let Z^n be the supermartingale which corresponds to $\bar{P}_n$ via (1.20). By Föllmer and Kramkov [1997], Lemma 5.2, there are processes

$$Z^{n,0} \in \ \mathrm{conv}(Z^n, Z^{n+1}, ...) \quad (n = 1, 2, ...)$$

and a right-continuous non-negative supermartingale Z such that $Z^{n,0}$ is Fatou convergent to Z on the set of rational points, i.e.,

$$Z_t = \limsup_{s\downarrow t, s\in\mathbb{Q}} \ \limsup_{n\to\infty} Z_s^{n,0} = \liminf_{s\downarrow t, s\in\mathbb{Q}} \ \liminf_{n\to\infty} Z_s^{n,0}$$

R-almost surely for $t \geq 0$. In particular $Z_T^{n,0}$ converges to Z_T R-almost surely because $Z^{n,0}$ is constant for $t \geq T$ for every $n \geq 1$. Furthermore, $VZ^{n,0}$ is Fatou convergent to the supermartingale VZ for every $V \in \mathcal{V}(x_0)$. Thus, part (ii) of Remark 1.2.6 shows that the probability measure $\bar{P}_0 := \bar{P}^Z$ belongs to $\bar{\mathcal{P}}$, and this completes the proof. □

Let us now formulate a general projection result for the class $\bar{\mathcal{P}}$ of extended martingale measures and for the class

$$\bar{\mathcal{Q}} := \{Q \times \delta_\infty : Q \in \mathcal{Q}\}.$$

Let the function f satisfy (1.17). In this case the definition (1.1) of $f(\cdot,\cdot)$ simplifies to

$$f(x,y) := \begin{cases} 0 & \text{if } y = 0, \\ yf\left(\frac{x}{y}\right) & \text{if } y > 0. \end{cases}$$

$f(\cdot,\cdot)$ is continuous on $(0,\infty) \times [0,\infty)$, and the f-divergence of $\bar{P} \in \bar{\mathcal{P}}$ with respect to $\bar{Q} \in \bar{\mathcal{Q}}$ is given by

$$f(\bar{P}|\bar{Q}) = E_Q\left[f\left(\frac{d(P^\infty)^a}{dQ}\right)\right] = E_Q\left[f\left(\frac{d(P^T)^a}{dQ}\right)\right] = f(P^T|Q)$$

due to Remark 1.0.2 and our assumption $\mathcal{F}_T = \mathcal{F}$, where $(P^\infty)^a$ and $(P^T)^a$ are the absolutely continuous parts of P^∞ and P^T with respect to Q. We will now consider the problem of minimizing $f(\bar{P}|\bar{Q})$ over $\bar{\mathcal{P}}$ and $\bar{\mathcal{Q}}$, and this is equivalent to minimizing $f(P^T|Q)$ over $\mathcal{P}^T$ and $\mathcal{Q}$.

Theorem 1.2.8. *Let $\mathcal{Q}$ be weakly compact in the sense of Assumption 1.2.1, and let the convex continuous function f satisfy Condition* (1.17). *Then there exist a robust f-projection $\bar{P}^*$ of $\bar{\mathcal{Q}}$ on $\bar{\mathcal{P}}$ and its reverse f-projection $\bar{Q}^*$, i.e.,*

$$f(\bar{P}^*|\bar{Q}^*) = f(\bar{\mathcal{P}}|\bar{\mathcal{Q}}) = \inf_{\bar{P}\in\bar{\mathcal{P}}} \inf_{\bar{Q}\in\bar{\mathcal{Q}}} f(\bar{P}|\bar{Q}) = \inf_{P^T\in\mathcal{P}^T} \inf_{Q\in\mathcal{Q}} f(P^T|Q).$$

Proof. Let $(Q_n)_{n\geq 1} \subseteq \mathcal{Q}$ and $(\bar{P}_n)_{n\geq 1} \subseteq \bar{\mathcal{P}}$ be such that $f(\bar{P}_n|\bar{Q}_n)$ converges to the infimum of the values $f(\bar{P}|\bar{Q})$ for $\bar{P} \in \bar{\mathcal{P}}$ and $Q \in \mathcal{Q}$, and define

$$\psi_n := \frac{dQ_n}{dR}.$$

By Delbaen and Schachermayer [1994], Lemma A1.1, we can choose

$$\psi_{n,0} \in \mathrm{conv}(\psi_n, \psi_{n+1}, ...) \quad (n = 1, 2, ...)$$

and a function ψ_0 such that

$$\psi_{n,0} \longrightarrow \psi_0 \qquad R-\text{almost surely}.$$

Since the set $\mathcal{K}_{\mathcal{Q}}$ is weakly compact, $\psi_0 \in \mathcal{K}_{\mathcal{Q}}$, i.e., ψ_0 is the density of some measure $Q^* \in \mathcal{Q}$. Due to Lemma 1.2.7, we can also choose

$$\bar{P}_{n,0} \in \ \mathrm{conv}(\bar{P}_n, \bar{P}_{n+1}, ...) \quad (n = 1, 2, ...)$$

and $\bar{P}^* \in \bar{\mathcal{P}}$ such that (1.22) holds.

Define $\phi_{n,0} := dP^T_{n,0}/dR\big|_{\mathcal{F}_T}$ and $\phi_0 := d(P^*)^T/dR\big|_{\mathcal{F}_T}$. Note first that

$$\begin{aligned} f(\bar{P}^*|\bar{Q}^*) &= E_R\left[f\left(\phi_0, \psi_0\right)\right] \\ &= E_R\left[\lim_{\epsilon\to 0} f\left(\phi_0+\epsilon, \psi_0\right)\right] \\ &= \lim_{\epsilon\to 0} E_R\left[f\left(\phi_0+\epsilon, \psi_0\right)\right] \end{aligned}$$

by monotone convergence, since $f(\cdot, y)$ is continuous and decreasing on $[0, \infty)$ and

$$E_R\left[f\left(\phi_0+\epsilon, \psi_0\right)\right] = E_{Q^*}\left[f\left(\frac{\phi_0+\epsilon}{\psi_0}\right)\right] \geq f(1+\epsilon) > -\infty$$

by Jensen's inequality. For any $\epsilon > 0$, it follows as in Schied and Wu [2005], Lemma 3.6, that the set $\{f^-(\phi+\epsilon, \psi) : \phi \in \mathcal{K}_{\bar{\mathcal{P}}},\ \psi \in \mathcal{K}_{\mathcal{Q}}\}$ is uniformly integrable, where $\mathcal{K}_{\bar{\mathcal{P}}} := \left\{dP^T/dR : \bar{P} \in \bar{\mathcal{P}}\right\}$. Indeed, if f is bounded from below by some constant b, then this follows from $f(\phi+\epsilon, \psi) \geq b\psi$ and the compactness of the set $\mathcal{K}_{\mathcal{Q}}$. Otherwise, let g denote the inverse of the the function $-f$ as in Kramkov and Schachermayer [1999], Lemma 3.4. Due to the assumption $f(\infty)/\infty = 0$, we have

$$\lim_{x\to\infty} \frac{g(x)}{x} = \lim_{y\to\infty} \frac{y}{-f(y)} = \infty.$$

Observe that

$$\begin{aligned} E_R\left[\psi g\left(f^-\left(\frac{\phi+\epsilon}{\psi}\right)\right)\right] &\le E_R\left[\psi g\left(-f\left(\frac{\phi+\epsilon}{\psi}\right)\right)\right] + g(0) \\ &= E_R[(\phi+\epsilon); \psi > 0] + g(0) \\ &\le 1 + \epsilon + g(0) =: M \end{aligned}$$

for all $\phi \in \mathcal{K}_{\bar{\mathcal{P}}}$ and $\psi \in \mathcal{K}_{\mathcal{Q}}$. Hence for every $a > 0$ there exists $c(a) > 0$ such that $g(x) \ge ax$ for all $x \ge c(a)$ and

$$E_R\left[f^-(\phi+\epsilon,\psi); f^-\left(\frac{\phi+\epsilon}{\psi}\right) \ge c(a)\right] \le \frac{1}{a} E_R\left[\psi g\left(f^-\left(\frac{\phi+\epsilon}{\psi}\right)\right)\right] \le \frac{M}{a}$$

for all $\phi \in \mathcal{K}_{\bar{\mathcal{P}}}$ and $\psi \in \mathcal{K}_{\mathcal{Q}}$. Now let $\delta > 0$ be given and take $c := c(2M/\delta)$. Then for $A \in \mathcal{F}$

$$\begin{aligned} E_R[f^-(\phi+\epsilon,\psi); A] &= E_R\left[f^-(\phi+\epsilon,\psi); A, f^-\left(\frac{\phi+\epsilon}{\psi}\right) \ge c\right] \\ &\quad + E_R\left[f^-(\phi+\epsilon,\psi); A, f^-\left(\frac{\phi+\epsilon}{\psi}\right) < c\right] \\ &\le \frac{\delta}{2} + cE_R[\psi; A]. \end{aligned}$$

Since $\mathcal{K}_{\mathcal{Q}}$ is uniformly integrable, we can find $\zeta > 0$ such that $cE_R[\psi; A] \le \delta/2$ as soon as $R(A) \le \zeta$, and we have completed the proof of the uniform integrability of the set $\{f^-(\phi+\epsilon,\psi) : \phi \in \mathcal{K}_{\bar{\mathcal{P}}},\ \psi \in \mathcal{K}_{\mathcal{Q}}\}$.

This implies now

$$\begin{aligned} E_R\left[f\left(\phi_0 + \epsilon, \psi_0\right)\right] &= E_R\left[\lim_{n\to\infty} f(\phi_{n,0}+\epsilon, \psi_{n,0})\right] \\ &= E_R\left[\lim_{n\to\infty} f^+(\phi_{n,0}+\epsilon, \psi_{n,0})\right] - E_R\left[\lim_{n\to\infty} f^-(\phi_{n,0}+\epsilon, \psi_{n,0})\right] \\ &\le \liminf_{n\to\infty} E_R[f(\phi_{n,0}+\epsilon, \psi_{n,0})] \\ &\le \liminf_{n\to\infty} E_R[f(\phi_{n,0}, \psi_{n,0})] \\ &\le \liminf_{n\to\infty} E_R[f(\phi_n, \psi_n)] = f(\bar{\mathcal{P}}|\bar{\mathcal{Q}}). \end{aligned}$$

The first equality follows from the continuity of $f(\cdot + \epsilon, \cdot)$ on $[0,\infty)^2$, the first inequality follows from Fatou's lemma (applied to the first term) and Lebesgue's theorem (applied to the second term) and the last one from the convexity of $f(\cdot,\cdot)$. This shows that $f(\cdot|\cdot)$ attains its minimum in $(\bar{P}^*, \bar{Q}^*)$. □

Remark 1.2.9. *Uniqueness of the density of the absolutely continuous part of $(P^*)^T$ with respect to Q^* holds as in Proposition 1.1.12 if the function f is strictly convex.*

1.3 Conclusion

We show the existence of a robust f-projection P^* of $\mathcal{Q}$ on $\mathcal{P}$ in the following two situations: (i) We have $f(\infty)/\infty = \infty$, the set of densities in $\mathcal{P}$ is closed, and the set of densities in $\mathcal{Q}$ is weakly compact; (ii) We have $f(\infty)/\infty = 0$, we minimize over the set of extended martingale measures $\bar{\mathcal{P}}$, and we assume that the set of densities in $\mathcal{Q}$ is weakly compact. In the first case the key idea is to apply Young's inequality in a suitable Orlizc space in order to show that a subset of $\mathcal{P}$ with bounded robust f-divergence is weakly compact. In the second case we start with $\mathcal{P}$ being the set of absolutely continuous martingale measures for some semimartingale and then extend this class in a suitable way in order to guarantee existence.

These existence results can now be applied when we solve the robust utility maximization problem, and this will be done in the next chapter.

Chapter 2

Robust Utility Maximization

In this chapter we solve the problem of maximizing a robust utility functional under a budget constraint. Let us consider an agent in a financial market who wants to determine a best possible payoff profile. In its general form, this problem of optimal portfolio choice consists of finding a maximal element X with respect to a given preference order $\succeq$ on some convex class of "affordable" contingent claims. Under mild conditions such a preference order admits a numerical representation

$$X \succeq \tilde{X} \quad \Longleftrightarrow \quad U(X) \geq U(\tilde{X})$$

in terms of some utility functional U. If it satisfies the axioms of von Neumann und Morgenstern [1944] or Savage [1954], U can be expressed in terms of a utility function $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ and a probability measure Q, i.e.,

$$U(X) = E_Q[u(X)]. \tag{2.1}$$

Here we use a "robust" extension of the expected utility approach, which was introduced by Gilboa and Schmeidler [1989]. Instead of a single probabilistic model $Q \ll R$, we take a whole class $\mathcal{Q}$ of such models and define the preference order $\succeq$ via the utility functional

$$U(X) := \inf_{Q \in \mathcal{Q}} E_Q[u(X)]. \tag{2.2}$$

Thus, model uncertainty is taken into account explicitly. As shown by Gilboa and Schmeidler [1989], such robust preferences can be characterized by certain behavioral axioms, and they resolve several well-known "paradoxa" which arise in the classical framework; see, for instance, Karni and Schmeidler [1991] or the book by Föllmer and Schied [2004], Chapter 2.5.

Our aim is now to maximize the robust utility functional (2.2) over some class of affordable contingent claims. Using a martingale approach we characterize the optimal contingent claim and prove its existence. To this end, we first have to solve the dual problem which consists of minimizing a certain f-divergence over the two sets of martingale and of subjective measures. The key idea is to characterize the robust f-projection and its reverse f-projection, which solve the dual problem, as certain worst case measures. This characterization goes back to results by Rüschendorf [1984], and it finally leads to the solution of the robust utility maximization problem. Analogously to Chapter 1 we distinguish between two types of utility functions: For those which are finite on the whole real line, the corresponding f-divergence in the dual problem satisfies the properties of Section 1.1: We have $\lim_{x\to\infty} f(x)/x = \infty$, and the existence result of Section 1.1 then leads to the existence of an optimal contingent claim. For those utility functions which are only defined on the positive halfline, we have $\lim_{x\to\infty} f(x)/x = 0$, hence we use the existence result of Section 1.2 in order to solve the utility maximization problem.

The set of affordable contingent claims will be defined in terms of expectations under certain martingale measures. This suggests that we have to formulate different budget constraints for the two types of utility functions in order to apply the existence results from the previous chapter. For the case of utility functions on the positive halfline, most of the work is already done in Section 1.2, and we will use the class of extended martingale measures in order to formulate the budget constraint. For utility functions that are finite on the whole real line, however, there is still some work to do. We have to cope with the difficulty of ensuring that the optimal claim indeed satisfies our budget constraint. To this end, we will use a characterization of f-projections, which requires a certain integrability condition. This excludes some of the measures and leads to a certain subset of martingale measures which defines the constraint.

This Chapter is organized as follows: In Section 2.1 we give the definition of the robust utility functional and introduce the convex conjugate of the utility function and the v_λ-divergence, that will define the dual problem. In Section 2.2 we present the solution to the utility maximization problem in the classical case $\mathcal{Q} = \{Q_0\}$ under the budget constraint of a complete market model. The main result is stated in Theorem 2.2.3. This will then be used to tackle the general problem in Section 2.3, where we first introduce the budget constraint and then solve the robust utility maximization problem. In Proposition 2.3.8 we characterize the measures that solve the dual problem as worst case measures. The solution to the utility maximization problem is presented in Theorems 2.3.9 and 2.3.10 for the two types of utility functions, respectively. In Section 2.4 we illustrate our approach by some examples and show how our methods allow us to easily solve a closely related problem: The one of expenditure minimization under the constraint of a minimum level of robust expected utility.

2.1 Preliminaries

We consider a market over a finite time horizon T that consists of d risky assets and a bond which is used for discounting. The discounted price processes of the stocks are given by an $\mathbb{R}^d$-valued semimartingale S on a filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, R)$, where the filtration is assumed to be right-continuous, $\mathcal{F}_0$ is trivial with respect to R and contains all sets with R-measure zero, and $\mathcal{F} = \mathcal{F}_T$. R is only considered as a reference measure, it does not represent our believe about the probability of market events. An $\mathcal{F}$-measurable random variable X will be interpreted as the value of a *financial position* or *contingent claim* with maturity T. Our aim is to determine some X^* that maximizes the robust utility functional $\inf_{Q\in\mathcal{Q}} E_Q[u(X)]$, where u is some utility function and $\mathcal{Q}$ is some set of *subjective* or *model measures.* The contingent claims that are considered in this maximization problem have to be affordable given some initial endowment. For the definition of affordability, we will work with the set of absolutely continuous martingale measures.

Definition 2.1.1. *A probability measure $P \ll R$ is called an* absolutely continuous martingale measure *if the discounted stock price process $(S_t)_{0\le t\le T}$ is a local martingale under P. If in addition $P \sim R$, then P is called an* equivalent martingale measure*. The class of absolutely continuous martingale measures will be denoted by $\mathcal{P}$, the class of equivalent martingale measures by $\mathcal{P}_e$.*

We assume that the set of equivalent martingale measures is non-empty, i.e.,

$$\mathcal{P}_e \neq \emptyset.$$

This assumption is equivalent to the absence of arbitrage opportunities; see Delbaen and Schachermayer [1994] and also Yan [1998] and [2005] for precise versions of this equivalence and for different choices of the numéraire which is used to define the discounted price process $(S_t)_{0\le t\le T}$.

In order to solve the robust utility maximization problem, we will apply the existence results from the previous chapter. The dual problem consists of minimizing an f-divergence over the set of subjective measures and a suitable set of martingale measures. We will see that in the case $\bar{x}_u = -\infty$

this set consists of all absolutely continuous martingale measures $\mathcal{P}$. In the case $\bar{x}_u = 0$ we have to consider the extended class of martingale measures $\bar{\mathcal{P}}$ from Definition 1.2.3. Since we want to treat both types of utility functions with $\bar{x}_u = -\infty$ and $\bar{x}_u = 0$ at the same time, let $\mathcal{P}'$ be the set of martingale measures in the first case, i.e.,

$$\mathcal{P}' := \mathcal{P} \quad \text{if } \bar{x}_u = -\infty, \tag{2.3}$$

and the set of projections P^T of extended martingale measures on the space $(\Omega, \mathcal{F})$ from Definition 1.2.3 in the second case, i.e.,

$$\mathcal{P}' := \{P^T : \bar{P} \in \bar{\mathcal{P}}\} \quad \text{if } \bar{x}_u = 0. \tag{2.4}$$

For simplicity, we just write $P \in \mathcal{P}'$, omitting the superscript T in the second case. Hence any $P \in \mathcal{P}'$ is a measure on our space $(\Omega, \mathcal{F})$ with $P \ll R$ and $P(\Omega) \leq 1$, but not necessarily a probability measure.

We will now first introduce the robust utility functional in Section 2.1.1, then the convex conjugate of the utility function in Section 2.1.2, and finally the v_λ-divergence in Section 2.1.3, which will be minimized in the dual problem.

2.1.1 The Robust Utility Functional

We assume that the set $\mathcal{Q}$ is convex, that all measures $Q \in \mathcal{Q}$ are absolutely continuous with respect to the reference measure R, and that

$$\forall\, \epsilon > 0\ \ \exists\, \delta > 0 \text{ such that } R(A) < \delta \implies Q(A) < \epsilon\ \ \forall\, Q \in \mathcal{Q}. \tag{2.5}$$

This can be interpreted as uniform absolute continuity of the set $\mathcal{Q}$ with respect to R. The subjective believes Q of the agent are not too different from the model that would be implied by the reference measure R: If an event is very likely or unlikely under R, then also under *all* $Q \in \mathcal{Q}$. Since, of course, the set of densities

$$\mathcal{K}_{\mathcal{Q}} := \left\{ \frac{dQ}{dR} : Q \in \mathcal{Q} \right\}$$

is bounded in $L^1(R)$, (2.5) corresponds to one of the equivalent definitions of uniform integrability; see Dellacherie and Meyer [1975], Theorem II.19.

Lemma 2.1.2. *Assumption* (2.5) *is satisfied if and only if the set $\mathcal{K}_{\mathcal{Q}}$ is uniformly integrable.*

Let $\bar{\mathcal{K}}_{\mathcal{Q}}$ be the $L^1(R)$-closure of the set $\mathcal{K}_{\mathcal{Q}}$.

Lemma 2.1.3. *$\bar{\mathcal{K}}_{\mathcal{Q}}$ is weakly compact.*

Proof. By Theorem 1.0.7 of Dellacherie and Meyer [1975], a uniformly integrable set in $L^1(R)$ is weakly relatively compact. Hence its closure if weakly compact. Note that since $\mathcal{K}_{\mathcal{Q}}$ is convex, the weak and the strong closure coincide due to Theorem 1.0.9 of Dunford and Schwartz [1958]. □

The set $\bar{\mathcal{K}}_{\mathcal{Q}}$ defines yet another set $\bar{\mathcal{Q}}$ of subjective measures by setting $Q(A) := E_R[Z; A]$ for $Z \in \bar{\mathcal{K}}_{\mathcal{Q}}$. Note that we have $\inf_{Q\in\mathcal{Q}} E_Q[u(X)] = \inf_{Q\in\bar{\mathcal{Q}}} E_Q[u(X)]$ if $u(X) \in L^1(Q)$ for all $Q \in \mathcal{Q}$ and $u(X)^- \in L^1(Q)$ for all $Q \in \bar{\mathcal{Q}}$.[1] Furthermore, $\bar{\mathcal{Q}}$ satisfies (2.5) and the Assumption 1.2.1 of weak compactness. In order to simplify the notations in the following, we will assume that $\mathcal{K}_{\mathcal{Q}}$ is already closed. Furthermore, a certain equivalence assumption will be needed.

Assumption 2.1.4. *We assume that the set $\mathcal{K}_{\mathcal{Q}}$ is weakly compact and that $\mathcal{Q}$ is equivalent to R in the sense that*

$$R(A) = 0 \quad \Longleftrightarrow \quad Q(A) = 0 \textit{ for all } Q \in \mathcal{Q}. \tag{2.6}$$

Let us now introduce our *utility function* u. We suppose that $u : \mathbb{R} \to \mathbb{R} \cup \{-\infty\}$ is strictly increasing, strictly concave, continuously differentiable in the interior of the essential domain $\mathrm{dom}(u) := \{x \in \mathbb{R} : u(x) > -\infty\}$, and satisfies the *Inada conditions*

$$u'(\infty) := \lim_{x\to\infty} u'(x) = 0, \tag{2.7}$$

$$u'(\bar{x}_u) := \lim_{x\searrow\bar{x}_u} u'(x) = \infty \tag{2.8}$$

for $\bar{x}_u := \inf\{x \in \mathbb{R} : u(x) > -\infty\}$. It follows that the interior of the essential domain of u is given by the open interval $(\bar{x}_u, \infty)$. Note that $\bar{x}_u$ might actually take the value $-\infty$. We assume that u has *regular asymptotic elasticity* (*RAE*) in the sense of Kramkov and Schachermayer [1999], Schachermayer [2001], and Frittelli and Rosazza Gianin [2004], i.e.,

$$\limsup_{x\to\infty} \frac{xu'(x)}{u(x)} < 1 \quad \text{ and, if } \bar{x}_u = -\infty, \quad \liminf_{x\to-\infty} \frac{xu'(x)}{u(x)} > 1. \tag{2.9}$$

[1] Let $Q_0 \in \bar{\mathcal{Q}}$, and let $(Q_n)_{n\geq 1}$ be a sequence in $\mathcal{Q}$ with $dQ_n/dR \to dQ_0/dR$ weakly in $L^1(R)$. We want to show that $E_{Q_0}[u(X)] \geq \inf_{Q\in\mathcal{Q}} E_Q[u(X)]$. If $E_{Q_0}[u(X)] = \infty$, the claim is obvious. Otherwise, define $u_m := (u \wedge m) \vee (-m)$ for $m \geq 1$. Then $|u_m(X)| \leq |u(X)| \in L^1(Q)$ for all $Q \in \bar{\mathcal{Q}}$. Then $\inf_{Q\in\mathcal{Q}} E_Q[u(X)] \leq \lim_{n\to\infty} E_{Q_n}[u(X)] = \lim_{m\to\infty} \lim_{n\to\infty} E_{Q_n}[u_m(X)] = \lim_{m\to\infty} E_{Q_0}[u_m(X)] = E_{Q_0}[u(X)]$ by Lebesgues's dominated convergence theorem.

Hence the marginal utility $u'(x)$ is less than the average utility $u(x)/x$ for large x and larger for small x.

Moreover we assume that

$$\bar{x}_u = 0 \qquad \text{or} \qquad \bar{x}_u = -\infty \tag{2.10}$$

and that

$$u(\infty) := \lim_{x\to\infty} u(x) = \infty \qquad \text{or} \qquad u(\infty) = 0. \tag{2.11}$$

In view of our optimization problem, this is no loss of generality since we can shift the origin along the two axes if necessary. Furthermore, if (2.9) holds for a utility function with $-\infty < \bar{x}_u \neq 0$, then it also holds for $\tilde{u}(x) := u(x+\bar{x}_u)$. Indeed $x\tilde{u}'(x)/\tilde{u}(x) = (x+\bar{x}_u)u'(x+\bar{x}_u)/u(x+\bar{x}_u) - \bar{x}_u u'(x+\bar{x}_u)/u(x+\bar{x}_u)$, and the last term converges to 0 as $x \to \infty$ due to (2.9). If (2.9) holds for a utility function with $0 \neq u(\infty) < \infty$, then it also holds for $\tilde{u}(x) := u(x) - u(\infty)$. Note that the first part of (2.9) holds since $\tilde{u} < 0$. If $\bar{x}_u = -\infty$, the second part holds due to $x\tilde{u}'(x)/\tilde{u}(x) = xu'(x)/u(x) \cdot u(x)/(u(x) - u(\infty))$, and the last factor converges to 1 as $x \to -\infty$.

Our aim is now to maximize the *robust utility functional*

$$U(X) := \inf_{Q\in\mathcal{Q}} E_Q[u(X)]. \tag{2.12}$$

Due to (2.6), a contingent claim X satisfies $U(X) > -\infty$ only if

$$X \geq \bar{x}_u \qquad R-\text{almost surely}, \tag{2.13}$$

and from now on we will only consider contingent claims with this property.

2.1.2 The Convex Conjugate Function

In order to connect the robust utility maximization problem to our discussion of robust f-projections, let us introduce the convex conjugate function $v : [0,\infty) \to \mathbb{R} \cup \{\infty\}$ of the concave utility function u:

$$v(y) := \sup_{x>\bar{x}_u} \{u(x) - xy\} = u(I(y)) - yI(y), \tag{2.14}$$

where $I := (u')^{-1} : (0,\infty) \to (\bar{x}_u, \infty)$ is decreasing from ∞ to $\bar{x}_u$ due to the Inada conditions (2.7) and (2.8). The second equality in (2.14) follows from basic calculus. Note that v is finite and differentiable with $v' = -I$ on $(0,\infty)$, and that $u(x) = \inf_{y>0}\{v(y) + xy\}$.

Let $v(\cdot,\cdot)$ be the corresponding convex function on $[0,\infty)^2$ defined as in (1.1) by

$$v(x,y) := \begin{cases} 0 & \text{if } x = y = 0, \\ x \lim\limits_{z\to\infty} \frac{v(z)}{z} & \text{if } y = 0, x > 0, \\ yv\left(\frac{x}{y}\right) & \text{if } y > 0. \end{cases} \tag{2.15}$$

Example 2.1.5. *Consider the following standard choices of a utility function u:*

(i) $u(x) = \log x$ *on* $(0,\infty)$ (logarithmic utility),

(ii) $u(x) = \frac{1}{\gamma}x^\gamma$ *on* $(0,\infty)$, $0 \neq \gamma \in (-\infty, 1)$ (power utility),

(iii) $u(x) = -\frac{1}{\alpha}e^{-\alpha x}$ *on* $\mathbb{R}$, $\alpha \in (0,\infty)$ (exponential utility).

The corresponding functions I and v are given by

(i) $I(x) = \frac{1}{x}, \quad v(x) = -\log x - 1,$

(ii) $I(x) = x^{1/(\gamma-1)}, \quad v(x) = \frac{1}{\beta}x^{-\beta}$ *for* $\beta = \frac{\gamma}{1-\gamma}$,

(iii) $I(x) = -\frac{1}{\alpha}\log x, \quad v(x) = \frac{x}{\alpha}(\log x - 1).$

(i) and (ii) belong to the type of utility functions that are only defined on the positive halfline, and the corresponding convex conjugate function satisfies $v(\infty)/\infty = 0$. On the other hand, the exponential utility function is finite on the whole real line, and the corresponding convex conjugate function satisfies $v(\infty)/\infty = \infty$.

When solving the utility maximization problem, we will distinguish between utility functions that are finite on the whole real line, i.e., with $\bar{x}_u = -\infty$, and utility functions whose slope converges to $-\infty$ for $x \to 0$, i.e., with $\bar{x}_u = 0$. These two types correspond to the two different f-divergences that we considered in Sections 1.1 and 1.2. This is shown in the following lemma along with other useful results.

Lemma 2.1.6. *Let the Inada conditions* (2.7) *and* (2.8) *hold.*

(i)

$$\frac{v(\infty)}{\infty} := \lim_{x\to\infty} \frac{v(x)}{x} = \lim_{x\to\infty} v'(x) = -\bar{x}_u. \tag{2.16}$$

(ii)

$$v(0) := \lim_{x \searrow 0} v(x) = u(\infty) := \lim_{x \to \infty} u(x).$$

(iii) The function $v(\cdot, \cdot)$ *is continuously differentiable on* $(0, \infty)^2$ *with derivatives*

$$\frac{\partial v(x,y)}{\partial x} = v'\left(\frac{x}{y}\right) = -I\left(\frac{x}{y}\right)$$

and

$$\frac{\partial v(x,y)}{\partial y} = v\left(\frac{x}{y}\right) - \frac{x}{y} v'\left(\frac{x}{y}\right) = u\left(I\left(\frac{x}{y}\right)\right).$$

(iv) Under the additional assumption (2.9) *of reasonable asymptotic elasticity, for any* $\lambda > 0$ *there are constants* $a(\lambda) > 0$ *and* $b(\lambda) \geq 0$ *such that*

$$v(\lambda y) \leq a(\lambda) v(y) + b(\lambda)(y+1). \tag{2.17}$$

Proof. (i) If $v(\infty) = \infty$ or $-\infty$, then we obtain from L'Hôpital's rule

$$\lim_{x \to \infty} \frac{v(x)}{x} = \lim_{x \to \infty} v'(x) = -\lim_{x \to \infty} (u')^{-1}(x) = -\bar{x}_u,$$

where the last equality follows from the Inada condition (2.8). $v(\infty)$ is finite if and only if $\lim_{x \to \infty} v'(x) = 0$, and this in turn is equivalent to $\bar{x}_u = 0$ due to $v'(x) = -(u')^{-1}(x)$ and the Inada condition (2.8).

(ii) On the one hand, we have

$$\begin{aligned}
\liminf_{x \searrow 0} v(x) &= \liminf_{x \searrow 0} \sup_{y > \bar{x}_u} \{u(y) - xy\} \\
&\geq \sup_{y > \bar{x}_u} \liminf_{x \searrow 0} \{u(y) - xy\} \\
&= \sup_{y > \bar{x}_u} u(y) \\
&= u(\infty).
\end{aligned}$$

On the other hand,

$$\limsup_{x \searrow 0} v(x) = \limsup_{x \searrow 0} \sup_{y > x} (u(I(y)) - yI(y)) \leq \limsup_{x \searrow 0} u(I(x)) = u(\infty)$$

since $yI(y) \geq 0$ if y is small enough and due to the Inada condition (2.7).

(iii) This is basic calculus.

(iv) follows from our Assumption (2.9) of regular asymptotic elasticity: For the case $\bar{x}_u = -\infty$, this was shown in Frittelli and Rosazza Gianin [2004], where results from Schachermayer [2001] were used. For the case $\bar{x}_u = 0$, note that $v' = -I \le 0$, hence v is decreasing, and (2.17) holds for $\lambda \ge 1$ with $a(\lambda) = 1$ and $b(\lambda) = 0$. By Corollary 6.1(iii) in Kramkov and Schachermayer [1999], there is $y_0 > 0$ and a function $a(\lambda) > 0$ such that $v(\lambda y) \le a(\lambda)v(y)$ for all $y < y_0$ and $\lambda < 1$. For $y \ge y_0$, we have $v(\lambda y) \le v(\lambda y_0) \le b(\lambda)(y+1)$, where $b(\lambda) := \max\{v(\lambda y_0)/y_0, 0\}$. Hence (2.17) holds also in this case. □

The definition (2.15) now reduces to

$$v(x,y) = \begin{cases} 0 & \text{if } x = y = 0, \\ -x\bar{x}_u & \text{if } y = 0, x > 0, \\ yv\left(\frac{x}{y}\right) & \text{if } y > 0. \end{cases}$$

2.1.3 The v_λ-Divergence

Our aim is now to define a suitable f-divergence that will give us the dual problem, which will consist of minimizing this f-divergence over the set $\mathcal{Q}$ of subjective measures and the set $\mathcal{P}'$ of martingale measures from the definitions (2.3) and (2.4), where $\mathcal{P}' = \mathcal{P}$ if $\bar{x}_u = -\infty$, and $\mathcal{P}' = \mathcal{P}^T$ if $\bar{x}_u = 0$.

We do not assume that the measures in $\mathcal{Q}$ are equivalent to the reference measure R such that all measures $P \in \mathcal{P}'$ would be absolutely continuous with respect to $Q \in \mathcal{Q}$. However, the densities of P with respect to Q play a crucial role in the solution of the utility maximization problem. For $P \in \mathcal{P}'$ and $Q \in \mathcal{Q}$ with densities $\phi := dP/dR$ and $\psi := dQ/dR$, we denote therefore by

$$\frac{dP}{dQ} := \frac{\phi}{\psi} \cdot 1_{\{\psi>0\}} + \infty \cdot 1_{\{\psi=0,\phi>0\}} \tag{2.18}$$

the generalized Radon-Nikodym density of P with respect to Q. Let v be the convex conjugate of the utility function u as defined in (2.14), define $v_\lambda(x) := v(\lambda x)$ for $\lambda > 0$, and denote by

$$\begin{aligned} v_\lambda(P|Q) :&= E_R\left[v\left(\lambda\frac{dP}{dR}, \frac{dQ}{dR}\right)\right] \\ &= E_Q\left[v\left(\lambda\frac{dP}{dQ}\right)\right] \\ &= E_Q\left[v\left(\lambda\frac{dP^a}{dQ}\right)\right] - \bar{x}_u\lambda P^s(\Omega) \\ &= v_\lambda(P^a|Q) - \bar{x}_u\lambda P^s(\Omega) \end{aligned} \tag{2.19}$$

the v_λ-*divergence* of $P \in \mathcal{P}'$ with respect to $Q \in \mathcal{Q}$, where P^a and P^s are the absolutely continuous and singular part in the Hahn-Lebesgue decomposition of P with respect to Q. For $\lambda = 1$, we simply write $v(P|Q)$. Recall from Definition 1.0.3 that P^* is called a v_λ-projection of Q on $\mathcal{P}$ if it minimizes $v_\lambda(P|Q)$ over the set $\mathcal{P}$, and it is a robust v_λ-projection of $\mathcal{Q}$ on $\mathcal{P}$ if it minimizes $\inf_{Q\in\mathcal{Q}} v_\lambda(P|Q)$. Furthermore, Q_P is called a reverse v_λ-projection of P on $\mathcal{Q}$ if it minimizes $v_\lambda(P|Q)$ over the set $\mathcal{Q}$.

Remark 2.1.7. *Due to Remark 1.0.2, the v_λ-divergence is well defined. If $\bar{x}_u = -\infty$ and P is not absolutely continuous with respect to Q, then $v_\lambda(P|Q) = \infty$. If $P \ll Q$, then $v_\lambda(P|Q) \geq v(\lambda)$ due to Jensen's inequality. If $\bar{x}_u = 0$, then v is decreasing, and $v_\lambda(P|Q) \geq v(\lambda P^a(\Omega)) \geq v(\lambda)$. Hence*

$$v_\lambda(P|Q) \geq v(\lambda)$$

in all cases. Note that $v(P|Q) < \infty$ implies $Q \ll P$ whenever $v(0) = u(\infty) = \infty$ and $P \ll Q$ whenever $\bar{x}_u = -\infty$.

Example 2.1.8. *Consider the three utility functions from Example 2.1.5. Let*

$$H(Q|P) := E_Q\left[\frac{dP}{dQ}\log\left(\frac{dP}{dQ}\right)\right]$$

denote the relative entropy of P with respect to Q. The corresponding divergences $v_\lambda(P|Q)$ are given by

(i) $H(Q|P) - (1 + \log\lambda)$,

(ii) $\frac{1}{\beta}\lambda^{-\beta}E_Q\left[\left(\frac{dQ}{dP}\right)^\beta\right]$ *for* $\beta = \frac{\gamma}{1-\gamma}$,

(iii) $\frac{\lambda}{\alpha}\left(H(P|Q) + \log\lambda - 1\right)$.

In particular $v_\lambda(P|Q) < \infty$ for all $\lambda > 0$ as soon as P and Q satisfy the corresponding conditions (i) $H(Q|P) < \infty$, (ii) $1/\beta\cdot E_Q\left[(dQ/dP)^\beta\right] < \infty$, *or* (iii) $H(P|Q) < \infty$. *Furthermore, in all three cases the robust v_λ-projections and its reverse projections are independent of λ.*

Remark 2.1.9. *Note that*

$$v(x,y) = \sup_{z>\bar{x}_u}\{yu(z) - xz\} = yu\left(I\left(\frac{x}{y}\right)\right) - xI\left(\frac{x}{y}\right)$$

for $x, y \geq 0$ with the convention $0/0 := 0$ and $x/0 := \infty$ for $x > 0$, and that the maximizer $z^* = I(x/y)$ of $yu(z) - xz$ is unique if $x > 0$ or $y > 0$. Furthermore, $v(0,0) = 0$. This leads to

$$v_\lambda(P|Q) = E_Q\left[u\left(I\left(\lambda\frac{dP}{dQ}\right)\right)\right] - E_P\left[I\left(\lambda\frac{dP}{dQ}\right)\right] \tag{2.20}$$

for any $\lambda > 0$, $P \in \mathcal{P}'$, and $Q \in \mathcal{Q}$. Furthermore, for any $X \geq \bar{x}_u$ we have on $\{dP/dR > 0\} \cup \{dQ/dR > 0\}$

$$\frac{dQ}{dR}u(X) - \lambda\frac{dP}{dR}X \leq v\left(\lambda\frac{dP}{dR}, \frac{dQ}{dR}\right)$$

with equality if and only if $X = I(\lambda dP/dQ)$ R-almost surely. On the remaining space we have zero on both sides. Hence for any $X \geq \bar{x}_u$ and any $\lambda > 0$,

$$E_R\left[\frac{dQ}{dR}u(X) - \lambda\frac{dP}{dR}X\right] \leq v_\lambda(P|Q)$$

with equality if and only if $X = I(\lambda dP/dQ)$ on $\{dP/dR > 0\} \cup \{dQ/dR > 0\}$ R-almost surely.

We will see that the solution to the utility maximization problem is of the form $I\left(\lambda dP/dQ\right)$, where the Radon-Nikodym density of P with respect to Q is defined in (2.18). This was already shown by many authors for the classical case with $\mathcal{Q} = \{Q_0\}$ and $P \ll Q_0$. Note that here

$$I\left(\lambda\frac{dP}{dQ}\right) = I\left(\lambda\frac{dP^a}{dQ}\right) \cdot 1_{A^c} + \bar{x}_u \cdot 1_A$$

and

$$E_P\left[I\left(\lambda\frac{dP}{dQ}\right)\right] = E_{P^a}\left[I\left(\lambda\frac{dP^a}{dQ}\right)\right] + \bar{x}_u P^s(\Omega),$$

where $A := \{dQ/dR = 0,\ dP/dR > 0\}$ is the support of the singular part of P. In particular $I(\lambda dP/dQ) = I(\lambda dP^a/dQ)$ R-almost surely if $\bar{x}_u = 0$, or if $\bar{x}_u = -\infty$ and $v(P|Q) < \infty$.

Let us now first solve a simplified utility maximization problem before we get to the general case in Section 2.3.

2.2 The Non-Robust Case in a "Complete Market" Setting

Let us fix $P \in \mathcal{P}'$ and $Q \in \mathcal{Q}$. We want to maximize the classical utility functional $E_Q[u(X)]$ over a set of affordable contingent claims. If $P \sim Q$, it is well known how to solve this classical problem; see, for instance, Karatzas and Shreve [1991]. Here we summarize the solution in a slightly more general form, which will then be extended to the robust case. Note that we only assume that $P \in \mathcal{P}'$, where $\mathcal{P}' := \mathcal{P}$ if $\bar{x}_u = -\infty$ and $\mathcal{P}' := \mathcal{P}^T$ if $\bar{x}_u = 0$.

Let $x_0 > \bar{x}_u$ be the agent's initial endowment. If the market is indeed complete with P being the unique equivalent martingale measure, then for any financial position $X \in L^1(P)$ there exists a trading strategy in the underlying assets S, described by a d-dimensional predictable, S-integrable process $(\xi_t)_{0 \le t \le T}$, such that P-almost surely

$$E_P[X|\mathcal{F}_t] = E_P[X] + \int_0^t \xi_s dS_s \qquad (0 \le t \le T),$$

i.e., any financial position is attainable by a self-financing strategy, see Jacod [1975], Theorem 5.4. Hence, the arbitrage-free price of a contingent claim $X \in L^1(P)$ is uniquely determined by the expectation $E_P[X]$. The usual budget constraint then is to require

$$E_P[X] \le x_0.$$

For $P \in \mathcal{P}'$ and $Q \in \mathcal{Q}$, we define the set of well defined and affordable contingent claims by

$$\mathcal{X}_{P,Q}(x_0) := \left\{ X \ge \bar{x}_u : X \in L^1(P),\ E_P[X] \le x_0,\ \text{and } u(X)^- \in L^1(Q) \right\}, \tag{2.21}$$

where the inequality $X \ge \bar{x}_u$ is meant in the R-almost sure sense. We now want to solve the problem

$$\text{Maximize } E_Q[u(X)] \text{ over all contingent claims } X \in \mathcal{X}_{P,Q}(x_0). \tag{2.22}$$

The following result will guarantee that under the condition $v(P|Q) < \infty$ a solution to the utility maximization problem (2.22) exists.

Lemma 2.2.1. *For $P \in \mathcal{P}'$ and $Q \in \mathcal{Q}$, the following conditions are equivalent:*

(i) $v(P|Q) < \infty$,

(ii) $v_\lambda(P|Q) < \infty$ *for any* $\lambda > 0$,

(iii) For any $\lambda > 0$, the contingent claim

$$X_\lambda := I\left(\lambda \frac{dP}{dQ}\right)$$

satisfies

$$X_\lambda \in L^1(P) \text{ and } u(X_\lambda) \in L^1(Q), \tag{2.23}$$

(iv) $X_\lambda^- \in L^1(P)$ *and* $u(X_\lambda)^+ \in L^1(Q)$ *for any* $\lambda > 0$.

Proof. The equivalence of (i) and (ii) follows from (2.17). In order to check the equivalence of (ii), (iii), and (iv), define $\rho := dP^a/dQ$ and note that (ii) is equivalent to $\bar{x}_u P^s(\Omega) > -\infty$ and $E_Q[v(\lambda\rho)] < \infty$ for any $\lambda > 0$ due to (2.19). For $0 < \lambda_1 < \lambda < \lambda_2$, the two estimates

$$v(\lambda_i \rho) \geq v(\lambda\rho) + v'(\lambda\rho)(\lambda_i - \lambda)\rho \qquad \text{on} \qquad \{0 < \rho < \infty\}$$

for $i = 1, 2$ show that $v'(\lambda\rho)\rho \in L^1(Q)$ and hence $I(\lambda\rho) = -v'(\lambda\rho) \in L^1(P^a)$ and $X_\lambda = I(\lambda\rho) + \bar{x}_u \cdot 1_A \in L^1(P)$, as soon as (ii) holds. Since $u(X_\lambda) = u(I(\lambda\rho))$ Q-almost surely and

$$u(I(\lambda\rho)) = v(\lambda\rho) + \lambda\rho I(\lambda\rho) \tag{2.24}$$

by (2.14), Condition (ii) also implies $u(X_\lambda) \in L^1(Q)$. Clearly, (iii) implies (iv). Conversely, (2.24) allows us to verify (ii) as soon as $u^+(X_\lambda) \in L^1(Q)$ and $X_\lambda^- \in L^1(P)$. Indeed, $v^-(\lambda\rho) \in L^1(Q)$ by convexity of v and $v^+(\lambda\rho) \leq u^+(I(\lambda\rho)) + \lambda\rho X_\lambda^-$. Moreover, if $\bar{x}_u = -\infty$, then $|\bar{x}_u| P^s(\Omega) \leq E_P[X_\lambda^-]$, hence $P \ll Q$ and $v_\lambda(P|Q) = E_Q[v(\lambda\rho)] < \infty$. □

The next lemma provides a method of ensuring that the optimal claim satisfies the budget constraint.

Lemma 2.2.2. *Suppose that $P \in \mathcal{P}'$ and $Q \in \mathcal{Q}$ are such that $v(P|Q) < \infty$ and let $x_0 > \bar{x}_u$. The function $h : (0, \infty) \to \mathbb{R}$ defined by*

$$h(\lambda) := v_\lambda(P|Q) + \lambda x_0$$

is strictly convex and continuously differentiable with derivative

$$h'(\lambda) = x_0 - E_P\left[I\left(\lambda \frac{dP}{dQ}\right)\right]. \tag{2.25}$$

In particular h attains its minimum in the unique value $\lambda_{P,Q} > 0$ such that

$$E_P\left[I\left(\lambda_{P,Q} \frac{dP}{dQ}\right)\right] = x_0. \tag{2.26}$$

Proof. The function $g(\lambda) := v(\lambda) + \lambda x_0$ is strictly convex and differentiable on $(0, \infty)$ with $g(0) = v(0) = u(\infty)$, $g' = x_0 - I$, $g'(0+) = -\infty$, and $\lim_{\lambda\to\infty} g'(\lambda) = x_0 - \bar{x}_u > 0$ due to (2.8), hence $\lim_{\lambda\to\infty} g(\lambda) = \infty$. In particular g is bounded from below. For $\rho := dP^a/dQ$, Jensen's inequality implies

$$\begin{aligned} h(\lambda) &= E_Q[v(\lambda\rho) + \lambda x_0 \rho] + x_0 \lambda P^s(\Omega) \\ &\geq E_Q[g(\lambda\rho)] \geq g(\lambda) \end{aligned}$$

since $P_s(\Omega) = 0$ if $\bar{x}_u = -\infty$ and $v(P|Q) < \infty$, and $x_0 > 0$ if $\bar{x}_u = 0$. Note that $g'(\lambda\rho)\,\rho \in L^1(Q)$ for any $\lambda > 0$ by Lemma 2.2.1. Using the monotonicity of g' in order to get an integrable bound, we can apply Fubini's theorem to conclude

$$\begin{aligned} h(\lambda_2) &= h(\lambda_1) + E_Q\left[\int_{\lambda_1}^{\lambda_2} g'(\lambda\rho)\,\rho d\lambda\right] + x_0 P^s(\Omega)(\lambda_2 - \lambda_1) \\ &= h(\lambda_1) + \int_{\lambda_1}^{\lambda_2} E_{P^a}\left[g'(\lambda\rho)\right] d\lambda + x_0 P^s(\Omega)(\lambda_2 - \lambda_1) \\ &= h(\lambda_1) + x_0(\lambda_2 - \lambda_1) - \int_{\lambda_1}^{\lambda_2} E_P\left[I\left(\lambda\frac{dP}{dQ}\right)\right] d\lambda, \end{aligned}$$

and this implies (2.25). Moreover, $h(\cdot)$ attains its unique minimum in some $\lambda := \lambda_{P,Q} > 0$ such that $h'(\lambda) = 0$ since h' is continuous by (2.23), $h(\infty) = g(\infty) = \infty$, and since (2.25) implies $h'(0+) = -\infty$ by monotone convergence and (2.7). Since I is strictly decreasing, the minimizing value $\lambda_{P,Q}$ is uniquely determined by the condition

$$E_P\left[I\left(\lambda\frac{dP}{dQ}\right)\right] = x_0.$$

□

The following theorem finally gives the solution to Problem (2.22).

Theorem 2.2.3. *Suppose that $P \in \mathcal{P}'$ and $Q \in \mathcal{Q}$ are such that $v(P|Q) < \infty$, and let $\lambda_{P,Q}$ be as in* (2.26). *The solution to Problem* (2.22) *is given by*

$$X_{P,Q} := I\left(\lambda_{P,Q}\frac{dP}{dQ}\right) \in L^1(P),$$

it is R-almost surely unique on the set $\{dP/dR > 0\} \cup \{dQ/dR > 0\}$, and the maximal expected utility is given by $v_{\lambda_{P,Q}}(P|Q) + \lambda_{P,Q}x_0$:

$$\begin{aligned} \max_{X \in \mathcal{X}_{P,Q}(x_0)} E_Q[u(X)] &= E_Q[u(X_{P,Q})] \\ &= v_{\lambda_{P,Q}}(P|Q) + \lambda_{P,Q} \cdot x_0 \qquad (2.27) \\ &= \min_{\lambda > 0}\left\{v_\lambda(P|Q) + \lambda x_0\right\}. \end{aligned}$$

Proof. Let $\phi = dP/dR$ and $\psi = dP/dR$. Any $X \in \mathcal{X}_{P,Q}(x_0)$ satisfies

$$\begin{aligned} E_Q[u(X)] &\leq E_Q[u(X)] + \lambda(x_0 - E_P[X]) \\ &= E_R\left[\psi u(X) - \lambda\phi X\right] + \lambda x_0 \\ &\leq E_R\left[v\left(\lambda\phi, \psi\right)\right] + \lambda x_0 \\ &= v_\lambda(P|Q) + \lambda x_0 \\ &= E_Q\left[u\left(I\left(\lambda\frac{dP}{dQ}\right)\right)\right] + \lambda\left(x_0 - E_P\left[I\left(\lambda\frac{dP}{dQ}\right)\right]\right) \end{aligned} \tag{2.28}$$

for any $\lambda > 0$, where we have used (2.20). Due to Lemma 2.2.2 and Remark 2.1.9, the two inequalities reduce to equalities if and only if $\lambda = \lambda_{P,Q}$ and $X = X_{P,Q}$ R-almost surely on $\{dP/dR > 0\} \cup \{dQ/dR > 0\}$, which shows that $X_{P,Q}$ is indeed the unique solution to Problem (2.22). □

With means of this solution to Problem (2.22) we are now able to solve the robust utility maximization problem in the general case.

2.3 The General Case

Our aim is now to solve the corresponding *robust* problem to (2.22) in an incomplete market model. We first determine a suitable constant λ^* similar to Lemma 2.2.2 and state one further assumption on the set $\mathcal{Q}$. Then we formulate the budget constraint in Section 2.3.1 and the robust utility maximization problem in Section 2.3.2. In Section 2.3.3 we finally solve the problem. We first give a characterization of the robust f-projection and its reverse f-projection in Proposition 2.3.8 and then formulate the solution to the robust utility maximization problem in Theorems 2.3.9 and 2.3.10.

From now on we assume that

$$v(\mathcal{P}'|\mathcal{Q}) = \inf_{P\in\mathcal{P}'} \inf_{Q\in\mathcal{Q}} v(P|Q) < \infty. \tag{2.29}$$

Due to the assumption (2.9) of reasonable asymptotic elasticity and Lemma 2.1.6(iv), this implies $v_\lambda(\mathcal{P}'|\mathcal{Q}) < \infty$ for all $\lambda > 0$.

Note that in the cases of Example 2.1.8 the v_λ-projection is in fact independent of λ. In general this does not hold true. Instead of Lemma 2.2.2, we now need the following result in order to identify a suitable constant λ^*.

Lemma 2.3.1. *Assume that $v(\mathcal{P}'|\mathcal{Q}) < \infty$. For $x_0 > \bar{x}_u$, the function*

$$H(\lambda) := v_\lambda(\mathcal{P}'|\mathcal{Q}) + \lambda x_0 = \inf_{P\in\mathcal{P}'} \inf_{Q\in\mathcal{Q}} v_\lambda(P|Q) + \lambda x_0$$

is convex and achieves its infimum in some finite value $\lambda^* > 0$.

Proof. Due to the reasonable asymptotic elasticity assumption (2.9) and Lemma 2.1.6(iv), H is finite. Let us first show that H is convex. Let $\lambda^1, \lambda^2 > 0$, and let $\epsilon > 0$ be fixed. Choose $P^i \in \mathcal{P}'$ and $Q^i \in \mathcal{Q}$ such that for $i = 1, 2$

$$H(\lambda^i) + \epsilon \geq v_{\lambda^i}(P^i|Q^i) + \lambda^i x_0.$$

Let $\alpha \in [0,1]$. Since the sets $\mathcal{P}'$ and $\mathcal{Q}$ are convex, we see that $\tilde{P} := \left(\alpha\lambda^1 P^1 + (1-\alpha)\lambda^2 P^2\right)/(\alpha\lambda^1 + (1-\alpha)\lambda^2) \in \mathcal{P}'$ and $\alpha Q^1 + (1-\alpha)Q^2 \in \mathcal{Q}$. Let p^i, $\tilde{p}$, and q^i be the corresponding densities with respect to R. Then

$$\begin{aligned}
&H(\alpha\lambda^1 + (1-\alpha)\lambda^2) \\
&\leq E_R\left[v\left((\alpha\lambda^1 + (1-\alpha)\lambda^2)\tilde{p},\ \alpha q^1 + (1-\alpha)q^2\right)\right] + (\alpha\lambda^1 + (1-\alpha)\lambda^2)x_0 \\
&= E_R\left[v\left(\alpha\lambda^1 p^1 + (1-\alpha)\lambda^2 p^2,\ \alpha q^1 + (1-\alpha)q^2\right)\right] + (\alpha\lambda^1 + (1-\alpha)\lambda^2)x_0 \\
&\leq E_R\left[\alpha v\left(\lambda^1 p^1,\ q^1\right) + (1-\alpha)v\left(\lambda^2 p^2,\ q^2\right)\right] + (\alpha\lambda^1 + (1-\alpha)\lambda^2)x_0 \\
&= \alpha\ \left(v_{\lambda^1}(P^1|Q^1) + \lambda_1 x_0\right) + (1-\alpha)\ \left(v_{\lambda^2}(P^2|Q^2) + \lambda_2 x_0\right) \\
&\leq \alpha\ H(\lambda^1) + (1-\alpha)\ H(\lambda^2) + \epsilon,
\end{aligned}$$

where the second inequality follows from the convexity of v. Since ϵ was arbitrary the proof of the convexity of X is complete.

It can be easily shown that H converges to infinity as λ goes to infinity: It follows from Remark 2.1.7 that $v_\lambda(P|Q) \geq v(\lambda)$. Hence

$$\lim_{\lambda\to\infty} H(\lambda) \geq \lim_{\lambda\to\infty}(v(\lambda) + \lambda x_0) = \lim_{\lambda\to\infty} \lambda\left(\frac{v(\lambda)}{\lambda} + x_0\right) = \infty,$$

where the last equality follows from (2.16) and the fact that $x_0 > \bar{x}_u$. This implies that the convex function H achieves its infimum in some finite value λ^*. It remains to show that $\lambda^* > 0$. It follows from Jensen's inequality as above that $H(0) := \lim_{\lambda\searrow 0} H(\lambda) \geq v(0)$. If $v(0) = u(\infty) = \infty$, it is obvious that $\lambda^* > 0$. Otherwise, assume that $\lambda^* = 0$. We can choose $P \in \mathcal{P}'$ and $Q \in \mathcal{Q}$ such that $v(P|Q) < \infty$, and then $v_\lambda(P|Q) < \infty$ for all $\lambda > 0$ due to Lemma 2.2.1. Hence for any $0 < \epsilon < \lambda$,

$$\begin{aligned}
v(0) \leq H(0) &\leq H(\lambda) \\
&\leq v_\lambda(P|Q) + \lambda x_0 \\
&\leq v_\epsilon(P|Q) + (\lambda - \epsilon)\left(x_0 - E_P\left[I\left(\lambda\frac{dP}{dQ}\right)\right]\right),
\end{aligned}$$

where the last inequality follows from the convexity of $v_\lambda(P|Q)$ in $\lambda \in (0,\infty)$ and (2.25). Since $v(\epsilon dP/dR, dQ/dR) \in L^1(R)$ for all $\epsilon \geq 0$, $v_\epsilon(P|Q)$ converges to $v(0)$ as $\epsilon \searrow 0$ by the dominated convergence theorem. This implies

$$E_P\left[I\left(\lambda\frac{dP}{dQ}\right)\right] \leq x_0$$

for all $\lambda > 0$. But due to Lemma 2.2.1 and the Inada condition (2.7) we can choose $\lambda > 0$ such that the expectation is larger than x_0, a contradiction. $\square$

Due to Theorems 1.1.2 and 1.2.8, there exists $(P^*, Q^*) \in \mathcal{P}' \times \mathcal{Q}$ that minimizes $v_{\lambda^*}(P|Q)$ over the sets $\mathcal{P}'$ and $\mathcal{Q}$. In order to solve the robust utility maximization problem, it will be useful to characterize the measures P^* and Q^* as certain worst case measures. To this end, we need one further assumption on the set $\mathcal{Q}$.

Assumption 2.3.2. *For $Q \in \mathcal{Q}$, define the measure $Q_\alpha := \alpha Q + (1-\alpha)Q^* \in \mathcal{Q}$. We assume that for any $Q \in \mathcal{Q}$ there exists $\alpha \in (0, 1]$ such that*

$$v(P^*|Q_\alpha) < \infty. \tag{2.30}$$

Remark 2.3.3. *(i) Note that Assumption* (2.30) *is equivalent to*

$$v_\lambda(P^*|Q_\alpha) < \infty \text{ for all } \lambda > 0 \tag{2.31}$$

due to our Assumption (2.9) *and Lemma 2.1.6(iv).*

(ii) If u is bounded from above as for the exponential utility function, then Assumption 2.3.2 is always satisfied. Indeed, take $Q \in \mathcal{Q}$ and define θ_0, θ, and θ_α as the densities of the absolutely continuous parts of Q^*, Q, and Q_α with respect to P^*. Recall from Remark 1.0.5 that

$$v(P^*|Q_\alpha) = \hat{v}(Q_\alpha|P^*) = E_{P^*}\left[\hat{v}(\theta_\alpha)\right] + \hat{v}'(\infty)Q^s_\alpha(\Omega)$$

for $\hat{v}(x) := xv(1/x)$. Note that

$$\hat{v}'(x) = v\left(\frac{1}{x}\right) + \frac{1}{x}I\left(\frac{1}{x}\right) = u\left(I\left(\frac{1}{x}\right)\right) \tag{2.32}$$

and that $\hat{v}'(\infty) := \lim_{x\to\infty} \hat{v}'(x) = v(0) = u(\infty)$. Since $\hat{v}$ is convex,

$$\begin{aligned}
\hat{v}(\theta_\alpha) &\leq \hat{v}(\theta_0) - \hat{v}'(\theta_\alpha)(\theta_0 - \theta_\alpha) \\
&\leq \hat{v}(\theta_0) - u\left(I\left(\frac{1}{(1-\alpha)\theta_0}\right)\right)\theta_0 + u(\infty)\theta_\alpha \qquad \text{on } \{0 < \theta_\alpha < \infty\} \\
&= \hat{v}(\theta_0) - u\left(I\left(\frac{1}{1-\alpha}\rho_0\right)\right)\theta_0 + u(\infty)\theta_\alpha,
\end{aligned}$$

where $\rho_0 = 1/\theta_0 = dP^*/dQ^*$ on $\{0 < \theta_\alpha < \infty\}$. Since $u\left(I\left(\lambda\rho_0\right)\right) \in L^1(Q^*)$ for any $\lambda > 0$ by Lemma 2.2.1, we obtain $\hat{v}(\theta_\alpha) \in L^1(P^*)$ and $\hat{v}(Q_\alpha|P^*) = E_{P^*}[\hat{v}(\theta_\alpha)] + u(\infty)Q^s_\alpha(\Omega) < \infty$ for any $\alpha \in (0, 1]$.

Let us now formulate the budget constraint.

2.3.1 The Budget Constraint

When deciding for a utility maximizing investment the agent has to restrict himself to contingent claims that he can afford, that is, given his *initial endowment* x_0, the agent has to make sure that he is able to buy the desired claim. In order to formulate the budget constraint, we consider the two types of utility functions with $\bar{x}_u = 0$ and $\bar{x}_u = -\infty$ separately.

Utility Functions on the Positive Halfline

Let us fix an initial wealth $x_0 > \bar{x}_u = 0$. As in Section 1.2, a self-financing portfolio with initial value x_0 is a d-dimensional predictable, S-integrable process $(\xi_t)_{0 \le t \le T}$ which specifies the amount of each asset in the portfolio. The corresponding value process of the portfolio is given by

$$V_t := x_0 + \int_0^t \xi_s dS_s \qquad (0 \le t \le T). \tag{2.33}$$

The family $\mathcal{V}(x_0)$ denotes all non-negative value processes of self-financing portfolios with initial value equal to x_0.

A contingent claim $X \ge 0$ is *affordable* if there is a self-financing portfolio $V \in \mathcal{V}(x_0)$ such that

$$V_T \ge X \quad R-\text{almost surely.} \tag{2.34}$$

The optional decomposition theorem states that this is equivalent to

$$\sup_{P \in \mathcal{P}_e} E_P[X] \le x_0;$$

this was shown by Kramkov [1996] for the case where the stock price process S is locally bounded an by Föllmer and Kabanov [1998] without this boundedness assumption.

In order to apply the existence result from Theorem 1.2.8 and hence show the existence of a solution to the utility maximization problem, we have to consider the enlarged class of martingale measures from Section 1.2. The following lemma is a justification of this procedure. Recall the notions of the default time ζ from page 29 and of the sets $\bar{\mathcal{P}}$ and $\mathcal{P}^T$ from Definition 1.2.3.

Lemma 2.3.4. *For a contingent claim $X \ge 0$, the following conditions are equivalent:*

(i) $\sup_{P \in \mathcal{P}_e} E_P[X] \le x_0$.

(ii) There exists a value process $V \in \mathcal{V}(x_0)$ such that $V_T \geq X$ R-almost surely.

(iii) The corresponding claim $\bar{X} := X \cdot 1_{\{\zeta > T\}}$ satisfies the constraint

$$\sup_{\bar{P} \in \bar{\mathcal{P}}} E_{\bar{P}}[\bar{X}] \leq x_0.$$

(iv) $\sup_{P^T \in \mathcal{P}^T} E_{P^T}[X] \leq x_0$.

Proof. The equivalence of (i) and (ii) is a key result in the theory of superhedging; see Kramkov [1996], Theorems 2.1 and 3.2, and Föllmer and Kabanov [1998], Theorem 1. To check that (ii) implies (iii) note that for any $V \in \mathcal{V}(x_0)$ the process $(\bar{V}_t)$ is a $\bar{P}$-supermartingale with $\bar{V}_T \geq \bar{X}$ $\bar{P}$-almost surely because $\bar{P}(\bar{V}_T \geq \bar{X}) = P^T(V_T \geq X)$ and $P^T \ll R$. Since $P \times \delta_\infty \in \bar{\mathcal{P}}$ for any $P \in \mathcal{P}_e$, (iii) implies (i). To show the equivalence of (iii) and (iv) just note that we have $E_{\bar{P}}[\bar{X}] = E_{P^T}[X]$ since $\bar{X} = X \cdot 1_{\{\zeta > T\}}$. $\square$

In the following we will consider contingent claims that satisfy the constraint

$$\sup_{P \in \mathcal{P}^T} E_P[X] \leq x_0, \tag{2.35}$$

omitting the superscript T for simplicity.

Utility Function on the Whole Real Line

In the case where $\bar{x}_u = -\infty$ we cannot stick to the requirement (2.35) that the superhedging price of the considered contingent claims is bounded by the initial endowment x_0. This is due to the need for a characterization of the robust v_{λ^*}-projection as a worst case measure, which will ensure that the optimal claim is indeed affordable. Instead, we define a certain subset of martingale measures under which the expectation of the claims has to be bounded.

In order to guarantee the existence of a solution, we need

Assumption 2.3.5. *We assume that the set of densities of martingale measures*

$$\mathcal{K}_{\mathcal{P}} := \left\{ \frac{dP}{dR} : P \in \mathcal{P} \right\}$$

is closed in $L^1(R)$.

Remark 2.3.6. *If the price process S is assumed to be locally bounded, the class $\mathcal{P}$ of absolutely continuous martingale measures is closed in the sense of Assumption 2.3.5 since their densities ϕ can be characterized by the conditions $E_R[\phi S_\tau] = S_0$ for stopping times $\tau \leq T$ such that $S_\tau \in L^\infty(R)$; see, for instance, Frittelli [2000] or Bellini and Frittelli [2002].*

By Lemma 2.3.1 there exists a minimizer λ^* of the convex function $v_\lambda(\mathcal{P}|\mathcal{Q}) + \lambda x_0$. Due to Theorem 1.1.2, there exists $(P^*, Q^*) \in \mathcal{P} \times \mathcal{Q}$ that minimizes $v_{\lambda^*}(P|Q)$ over the sets $\mathcal{P}$ and $\mathcal{Q}$. For $P \in \mathcal{P}$, define the measure $P_\alpha := \alpha P + (1-\alpha)P^* \in \mathcal{P}$. Let the subset $\mathcal{P}_0$ of martingale measures be defined by

$$\begin{aligned} \mathcal{P}_0 :&= \{P \in \mathcal{P} : v(P_\alpha|Q^*) < \infty \text{ for some } \alpha \in (0,1]\} \\ &\supseteq \{P \in \mathcal{P} : v(P|Q^*) < \infty\}\,. \end{aligned} \tag{2.36}$$

Let us introduce a new concept of affordability.

Definition 2.3.7. *Let us say that X is* affordable with limited price *if there exist some $P \in \mathcal{P}_e$ such that $X \in L^1(P)$ and a trading strategy in the underlying assets, described by a d-dimensional predictable and S-integrable process $(\xi_t)_{0\leq t\leq T}$, such that the corresponding value process*

$$V_t := x_0 + \int_0^t \xi_s dS_s \qquad (0 \leq t \leq T) \tag{2.37}$$

satisfies

$$V_t \geq E_P[X|\mathcal{F}_t] \qquad (0 \leq t \leq T) \tag{2.38}$$

and in particular $V_T \geq X$ R-almost surely. If $\mathcal{P}_0 \cap \mathcal{P}_e \neq \emptyset$, we will say that the strategy has $\mathcal{P}_0$-limited price *if $X \in L^1(P)$ and* (2.38) *holds for any $P \in \mathcal{P}_0$.*

Note that for any $P \in \mathcal{P}_0$ we have $E_P[X] \leq V_0 = x_0$ due to (2.38). This implies the constraint

$$\sup_{P \in \mathcal{P}_0} E_P[X] \leq x_0 \tag{2.39}$$

for any contingent claim X which is affordable with $\mathcal{P}_0$-limited price. We will work with this constraint in the following. It will imply that the optimal claim X^* is affordable with $\mathcal{P}_0$-limited price and in addition attainable by some self-financing strategy ξ^*, that is,

$$X^* = x_0 + \int_0^T \xi_s^* dX_s.$$

2.3.2 The Problem

We are now ready to formulate the utility maximization problem. Let us fix an initial endowment $x_0 > \bar{x}_u$ and define the set of affordable contingent claims by

$$\mathcal{X}(x_0) := \Big\{ X \geq \bar{x}_u : \ X \in L^1(P) \text{ for all } P \in \mathcal{P}_0, \ \sup_{P \in \mathcal{P}_0} E_P[X] \leq x_0, \\ \text{and } u(X)^- \in L^1(Q) \text{ for all } Q \in \mathcal{Q} \Big\}$$

if $\bar{x}_u = -\infty$, and

$$\mathcal{X}(x_0) := \Big\{ X \geq \bar{x}_u : X \in L^1(P) \text{ for all } P \in \mathcal{P}^T, \ \sup_{P \in \mathcal{P}^T} E_P[X] \leq x_0, \\ \text{and } u(X)^- \in L^1(Q) \text{ for all } Q \in \mathcal{Q} \Big\}$$

if $\bar{x}_u = 0$. The requirement $X \geq \bar{x}_u$ is meant in the R-almost sure sense.

Our aim is now to find a contingent claim X^* that solves the problem

$$\text{Maximize } \inf_{Q \in \mathcal{Q}} E_Q[u(X)] \text{ over all contingent claims } X \in \mathcal{X}(x_0). \tag{2.40}$$

2.3.3 The Solution

Let λ^* be a minimizer of the convex function $v_\lambda(\mathcal{P}'|\mathcal{Q}) + \lambda x_0$, and let P^* be a robust v_{λ^*}-projection of $\mathcal{Q}$ on $\mathcal{P}'$, and let Q^* be the reverse v_{λ^*}-projection of P^* on $\mathcal{Q}$. The following Lemma extends the arguments in Goll and Rüschendorf [2001], Theorem 5.1, which go back to Theorem 5 by Rüschendorf [1984]. Together with our main Theorems 2.3.9 and 2.3.10 it will show that the measures P^* and Q^* are worst case measures.

Proposition 2.3.8. *Let the set $\mathcal{Q}$ satisfy Assumption 2.3.2 and assume that* (2.29) *holds. Define*

$$X^* := I\left(\lambda^* \frac{dP^*}{dQ^*}\right). \tag{2.41}$$

Then X^* has the following properties:

(i)

$$u(X^*) \in L^1(Q) \textit{ for all } Q \in \mathcal{Q},$$

and

$$E_{Q^*}[u(X^*)] = \min_{Q \in \mathcal{Q}} E_Q[u(X^*)]. \tag{2.42}$$

(ii) If $\bar{x}_u = -\infty$, then

$$X^* := I\left(\lambda^* \frac{dP^*}{dQ^*}\right) \in L^1(P) \text{ for all } P \in \mathcal{P}_0, \tag{2.43}$$

and

$$E_{P^*}[X^*] = \max_{P \in \mathcal{P}_0} E_P[X^*]. \tag{2.44}$$

If $P \sim Q^*$ for some $P \in \mathcal{P}_0$, then $P^* \sim Q^*$. If in addition $Q^* \sim R$, then for all $t \in [0,T]$ and $P \in \mathcal{P}_0$,

$$E_{P^*}[X^*|\mathcal{F}_t] \geq E_P[X^*|\mathcal{F}_t] \qquad R - \text{almost surely.} \tag{2.45}$$

(iii) If $\bar{x}_u = 0$, then

$$X^* := I\left(\lambda^* \frac{dP^*}{dQ^*}\right) \in L^1(P) \text{ for all } P \in \mathcal{P}^T, \tag{2.46}$$

and

$$E_{P^*}[X^*] = \max_{P \in \mathcal{P}^T} E_P[X^*]. \tag{2.47}$$

Proof. We will show the results for the cases $\bar{x}_u = -\infty$ and $\bar{x}_u = 0$ simultaneously. In order to simplify the notations, define $f := v_{\lambda^*}$. Let $\mathcal{P}_0' := \mathcal{P}_0$ if $\bar{x}_u = -\infty$ and $\mathcal{P}_0' := \mathcal{P}^T$ if $\bar{x}_u = 0$. Take $P \in \mathcal{P}_0'$ and define $P_\alpha := \alpha P + (1-\alpha)P^*$, $\rho := dP^a/dQ^*$, and $\rho_0 := d(P^*)^a/dQ^*$, and $\rho_\alpha := dP_\alpha^a/dQ^*$ for $\alpha \in (0,1)$. Note that in the case $\bar{x}_u = 0$ for any $P \in \mathcal{P}_0'$ there is $\alpha \in (0,1]$ such that $f(P_\alpha|Q^*) < \infty$. Indeed, since $f = v_{\lambda^*}$ is convex with derivative $f'(x) = -\lambda^* I(\lambda^* x) \leq -\lambda^* \bar{x}_u = 0$, we have

$$\begin{aligned} f(\rho_\alpha) &\leq f(\rho_0) - f'(\rho_\alpha)(\rho_0 - \rho_\alpha) \\ &\leq f(\rho_0) + \lambda^* I(\lambda^* \rho_\alpha)\rho_0 \qquad \text{on } \{0 < \rho_\alpha < \infty\} \\ &\leq f(\rho_0) + \lambda^* I(\lambda^*(1-\alpha)\rho_0)\rho_0. \end{aligned}$$

Since $\rho_0 I(\lambda\rho_0) \in L^1(Q^*)$ for any $\lambda > 0$ by Lemma 2.2.1, we obtain $f(\rho_\alpha) \in L^1(Q^*)$ for any $\alpha \in (0,1)$, hence $f(P_\alpha|Q^*) = E_{Q^*}[f(\rho_\alpha)] < \infty$.

Due to our assumption $\bar{x}_u = 0$ or $\bar{x}_u = -\infty$, we have $f(P|Q^*) = f(P^a|Q^*)$ if $f(P|Q^*) < \infty$. Since P^* is an f-projection of Q^* on $\mathcal{P}'$ and $f := v_{\lambda^*}$ is differentiable on $(0,\infty)$, a criterion in Rüschendorf [1984], Theorem 5, for f-projections implies

$$E_{Q^*}[f'(\rho_0)(\rho - \rho_0)] \geq 0. \tag{2.48}$$

For the convenience of the reader we include the argument: The function $\alpha \mapsto f(\rho_\alpha)$ is convex on $[0,1]$, and so

$$Z_\alpha := \frac{f(\rho_\alpha) - f(\rho_0)}{\alpha}$$

is increasing in α and decreasing to $Z_0 = f'(\rho_0)(\rho - \rho_0)$ as $\alpha \searrow 0$. By definition of $\mathcal{P}_0$ in the case $\bar{x}_u = -\infty$ and the first paragraph of this proof for the case $\bar{x}_u = 0$, there is $\alpha_0 \in (0,1]$ such that $Z_{\alpha_0} \in L^1(Q^*)$, and Z_α is bounded by Z_{α_0} for $\alpha \le \alpha_0$. By monotone convergence we obtain $Z_0 \in L^1(Q^*)$ and $E_{Q^*}[Z_0] \ge 0$, since $E_{Q^*}[Z_\alpha] = \alpha^{-1}\left(f(P_\alpha|Q^*) - f(P^*|Q^*)\right) \ge 0$ for any $\alpha > 0$. This completes the proof of (2.48).

Recall that $X^* = I(\lambda^* \rho_0)$ since $\bar{x}_u = 0$ or, if $\bar{x}_u = -\infty$, $P^* \ll Q^*$. In our situation we have $f'(x) = -\lambda^* I(\lambda^* x)$ and $f'(\rho_0)\rho_0 \in L^1(Q^*)$ by Lemma 2.2.1, hence $f'(\rho_0)\rho \in L^1(Q^*)$ and $X^* \in L^1(P)$ since $Z_0 \in L^1(Q^*)$. Hence (2.43) and (2.46) are proven. Moreover, Inequality (2.48) allows us to conclude

$$E_P\left[f'\left(\frac{dP^*}{dQ^*}\right)\right] \ge E_{P^*}\left[f'\left(\frac{dP^*}{dQ^*}\right)\right], \tag{2.49}$$

and this amounts to (2.44) and (2.47).

In order to verify (2.42), take $\hat{f}(x) := xf(1/x)$. Then Q^* is the $\hat{f}$-projection of P^* on $\mathcal{Q}$, and $\hat{f}'(dQ^*/dP^*) = u(X^*)$ by (2.32). Note that due to our assumption $u(\infty) = 0$ or $u(\infty) = \infty$ we have $f(P^*|Q) = f(P^*|Q^a)$ for any $Q \in \mathcal{Q}$ with $f(P^*|Q) < \infty$. Q^*-integrability of $u(X^*)$ follows from Lemma 2.2.1. Now we apply the argument above in terms of $\hat{f}$, reversing the role of the sets $\mathcal{Q}$ and $\mathcal{P}$ to obtain

$$E_Q[u(X^*)] \ge E_{Q^*}[u(X^*)]$$

due to Assumption 2.3.2. Q-integrability of $u(X^*)$ for $Q \in \mathcal{Q}$ follows as above.

In order to show that $P^* \sim Q^*$ if there is $P \in \mathcal{P}_0$ with $P \sim Q^*$, take $P \in \mathcal{P}_0$ with $P \sim Q^*$. The integrability result (2.43) implies $E_P[I(\lambda^* dP^*/dQ^*)] < \infty$. Due to the Inada condition (2.7) we thus have $P\left(dP^*/dQ^* = 0\right) = 0$ and hence $Q^*\left(dP^*/dQ^* = 0\right) = 0$. But this means that $Q^* \ll P^*$. On the other hand, since $\bar{x}_u = -\infty$, $f(P^*|Q^*) < \infty$ implies $P^* \ll Q^*$.

Let us finally show the conditional estimate (2.45). For $P \in \mathcal{P}_0$ and $t \in (0,T)$, we write $\rho_0 = \rho_{0,t}\hat{\rho}_{0,t}$ where $\rho_{0,t} := d(P^*)^a/dQ^*|_{\mathcal{F}_t}$ and $\hat{\rho}_{0,t}$ is the conditional density with respect to $\mathcal{F}_t$. In the same way we define ρ_t, $\hat{\rho}_t$, $\rho_{\alpha,t}$

and $\hat{\rho}_{\alpha,t}$. Due to (2.17), we have on $\{\rho_{\alpha,t} > 0\}$

$$\begin{aligned} f(\rho_{0,t}\hat{\rho}_{\alpha,t}) &= f\left(\frac{\rho_{0,t}}{\rho_{\alpha,t}}\rho_{\alpha,t}\hat{\rho}_{\alpha,t}\right) \\ &\leq a\left(\frac{\rho_{0,t}}{\rho_{\alpha,t}}\right) f(\rho_{\alpha,t}\hat{\rho}_{\alpha,t}) + b\left(\frac{\rho_{0,t}}{\rho_{\alpha,t}}\right)(\rho_{\alpha,t}\hat{\rho}_{\alpha,t}+1) \\ &= a\left(\frac{\rho_{0,t}}{\rho_{\alpha,t}}\right) f(\rho_{\alpha}) + b\left(\frac{\rho_{0,t}}{\rho_{\alpha,t}}\right)(\rho_{\alpha}+1). \end{aligned}$$

For $\alpha \in (0, \alpha_0]$, we have $E_{Q^*}[f(\rho_\alpha)|\mathcal{F}_t] < \infty$ Q^*-almost surely, and this implies that also $E_{Q^*}[f(\rho_{0,t}\hat{\rho}_{\alpha,t})|\mathcal{F}_t] < \infty$ Q^*-almost surely on $\{\rho_{\alpha,t} > 0\}$. If $f(0) = 0$, then $f(\rho_{0,t}\hat{\rho}_{\alpha,t}) = 0$ on $\{\rho_{\alpha,t} = 0\}$ due to the definition of $\rho_{\alpha,t}$. If $f(0) = \infty$, then $\rho_{\alpha,t} > 0$ R-almost surely since $f(P_\alpha|Q^*) < \infty$. Hence $E_{Q^*}[f(\rho_{0,t}\hat{\rho}_{\alpha,t})|\mathcal{F}_t] < \infty$ Q^*-almost surely on Ω. Furthermore, we can show that

$$E_{Q^*}[f(\rho_{0,t}\hat{\rho}_{\alpha,t})|\mathcal{F}_t] \geq E_{Q^*}[f(\rho_{0,t}\hat{\rho}_{0,t})|\mathcal{F}_t] \qquad Q^* - \text{almost surely.}$$

Indeed, the measure $\tilde{P}$ with density

$$\tilde{\rho} := \begin{cases} \rho_{0,t}\hat{\rho}_{\alpha,t} & \text{on } A, \\ \rho_0 & \text{on } A^c, \end{cases}$$

where $A := \{E_{Q^*}[f(\rho_{0,t}\hat{\rho}_{\alpha,t})|\mathcal{F}_t] < E_{Q^*}[f(\rho_{0,t}\hat{\rho}_{0,t})|\mathcal{F}_t]\}$ belongs to $\mathcal{P}$, and $Q^*(A) > 0$ would imply

$$f(\tilde{P}|Q^*) = E_{Q^*}[f(\tilde{\rho})] = E_{Q^*}[E_{Q^*}[f(\tilde{\rho})|\mathcal{F}_t]] < E_{Q^*}[f(\rho_0)] = f(P^*|Q^*)$$

which contradicts the minimality of P^*. We can now repeat the argument above, with

$$Z_{\alpha,t} := \frac{f(\rho_{0,t}\hat{\rho}_{\alpha,t}) - f(\rho_0)}{\alpha}$$

instead of Z_α, to obtain

$$\rho_{0,t}E_{Q^*}[f'(\rho_0)(\hat{\rho}_t - \hat{\rho}_{0,t})|\mathcal{F}_t] \geq 0 \qquad Q^* - \text{almost surely.}$$

Since $Q^* \sim P^* \sim R$ and hence $\rho_{0,t} > 0$ R-almost surely, the proof of (2.45) is complete. □

Let us now show how the existence of a robust v_{λ^*}-projection P^* of $\mathcal{Q}$ on $\mathcal{P}'$ together with the characterization of P^* and its reverse v_{λ^*}-projection Q^* in Proposition 2.3.8 yields the solution of the optimization problem (2.40). We distinguish again between the two cases $\bar{x}_u = -\infty$ and $\bar{x}_u = 0$.

Theorem 2.3.9. *Let $\bar{x}_u = -\infty$, and assume that* (2.29) *holds. Furthermore, let the set $\mathcal{P}$ satisfy Assumption 2.3.5, and let $\mathcal{Q}$ satisfy Assumptions 2.1.4 and 2.3.2. Then the robust utility maximization problem* (2.40) *has the solution*

$$X^* := I\left(\lambda^* \frac{dP^*}{dQ^*}\right), \tag{2.50}$$

and the maximizer is R-almost surely unique on the set $\{dP^*/dR > 0\} \cup \{dQ^*/dR > 0\}$. The maximal value of the robust utility is given by

$$\inf_{Q \in \mathcal{Q}} E_Q[u(X^*)] = E_{Q^*}[u(X^*)] = v_{\lambda^*}(\mathcal{P}|\mathcal{Q}) + \lambda^* x_0.$$

The robust problem (2.40) *in an incomplete market is equivalent to the classical problem* (2.22) *under the measures P^* and Q^*. Moreover, the contingent claim X^* is attainable by some self-financing strategy, and it is affordable with $\mathcal{P}_0$-limited price if $P^* \sim Q^* \sim R$.*

Theorem 2.3.10. *Let $\bar{x}_u = 0$, and assume that* (2.29) *holds. Furthermore, let the set $\mathcal{Q}$ satisfy Assumptions 2.1.4 and 2.3.2. Then the robust utility maximization problem* (2.40) *has the solution*

$$X^* := I\left(\lambda^* \frac{dP^*}{dQ^*}\right), \tag{2.51}$$

which is R-almost surely unique on the set $\{dP^*/dR > 0\} \cup \{dQ^*/dR > 0\}$, and the maximal value of the robust utility is given by

$$\inf_{Q \in \mathcal{Q}} E_Q[u(X^*)] = E_{Q^*}[u(X^*)] = v_{\lambda^*}(\mathcal{P}^T|\mathcal{Q}) + \lambda^* x_0.$$

The robust problem (2.40) *in an incomplete market is equivalent to the classical problem* (2.22) *under the measures P^* and Q^*.*

Proof of Theorem 2.3.9. If $\bar{x}_u = -\infty$, then $\lim_{x\to\infty} v(x)/x = -\bar{x}_u = \infty$, and the same property holds for $f = v_{\lambda^*}$. Hence by Theorem 1.1.2 there exist $P^* \in \mathcal{P}$ and $Q^* \in \mathcal{Q}$ that minimize the v_{λ^*}-divergence over the sets $\mathcal{P}$ and $\mathcal{Q}$.

For any $X \in \mathcal{X}(x_0)$, the estimate (2.28) applied to P^*, Q^*, and $\lambda > 0$ shows that

$$\begin{aligned} U(X) = \inf_{Q \in \mathcal{Q}} E_Q[u(X)] &\le E_{Q^*}[u(X)] \\ &\le \inf_{\lambda \ge 0} \{v_\lambda(P^*|Q^*) + \lambda x_0\} \\ &= v_{\lambda^*}(P^*|Q^*) + \lambda^* x_0 \\ &= E_{Q^*}\left[u\left(X^*\right)\right] + \lambda^*(x_0 - E_{P^*}[X^*]). \end{aligned}$$

Note that the function $\lambda \mapsto v_\lambda(P^*|Q^*) + \lambda x_0$ attains its minimum in λ^*. Thus, Lemma 2.2.2 implies that $E_{P^*}[X^*] = x_0$, and this yields

$$\begin{aligned} U(X) &\leq v_{\lambda^*}(P^*|Q^*) + \lambda^* x_0 \\ &= E_{Q^*}\left[u\left(X^*\right)\right]. \end{aligned}$$

Lemma 2.3.8 shows that $X^* \in \mathcal{X}(x_0)$, that X^* satisfies the budget constraint (2.39), and that

$$E_{Q^*}\left[u\left(X^*\right)\right] = \min_{Q\in\mathcal{Q}} E_Q\left[u\left(X^*\right)\right].$$

This concludes the proof that X^* is optimal, with $U(X^*) = v_{\lambda^*}(P^*|Q^*)+\lambda^* x_0$. Furthermore, X^* is the solution to Problem (2.22) under P^* and Q^*.

In order to show uniqueness, assume that $\tilde{X} \in \mathcal{X}(x_0)$ solves Problem (2.40). Then we have $E_{P^*}[\tilde{X}] \leq x_0$ and hence

$$\inf_{Q\in\mathcal{Q}} E_Q[u(\tilde{X})] \leq E_{Q^*}[u(\tilde{X})] \leq E_{Q^*}[u(X^*)].$$

The second inequality holds strictly unless $\tilde{X} = X^*$ R-almost surely on $\{dP^*/dR > 0\} \cup \{dQ^*/dR > 0\}$. This follows from the fact that X^* is the solution to Problem (2.22) under P^* and Q^* and from the uniqueness result in Theorem 2.2.3. But the strict inequality is a contradiction to $E_{Q^*}[u(X^*)] = \inf_{Q\in\mathcal{Q}} E_Q[u(X^*)]$. Thus $\tilde{X} = X^*$ R-almost surely on $\{dP^*/dR > 0\} \cup \{dQ^*/dR > 0\}$.

Moreover, we obtain from Goll and Rüschendorf [2001], Theorem 3.2, that

$$X^* = x_0 + \int_0^T \xi_s dS_s \tag{2.52}$$

for some trading strategy $(\xi_t)_{0\leq t\leq T}$ such that the corresponding value process $V_t := x_0 + \int_0^t \xi_s dS_s$ $(0 \leq t \leq T)$ is a P^*-martingale; this representation is based on results by Yor [1978] and Jacod [1979]. For any $P \in \mathcal{P}_0$, the conditional estimates (2.45) together with (2.52) show that

$$V_t = E_{P^*}[X|\mathcal{F}_t] \geq E_P[X|\mathcal{F}_t] \qquad (0 \leq t \leq T)$$

R-almost surely if $P^* \sim Q^* \sim R$. Thus, X^* is affordable with $\mathcal{P}_0$-limited price. □

Proof of Theorem 2.3.10. If $\bar{x}_u = 0$, then by Theorem 1.2.8 there exist $P^* \in \mathcal{P}^T$ and $Q^* \in \mathcal{Q}$ that minimize the v_{λ^*}-divergence over the sets $\mathcal{P}^T$ and $\mathcal{Q}$. Now the claims are shown in as in the previous proof replacing the sets $\mathcal{P}_0$ and $\mathcal{P}$ by $\mathcal{P}^T$. □

These theorems provide a complete solution to the utility maximization problem (2.40). The solution is of the same form (2.50) as in the classical case with $\mathcal{Q} = \{Q_0\}$. If $\bar{x}_u = 0$, then there exists a self-financing strategy whose portfolio process dominates the optimal contingent claim at the final time T; see Lemma 2.3.4. If P^* is indeed an equivalent martingale measure, then the optimal claim is even attainable by some self-financing strategy. This follows from Proposition 2.3.8(iii) and Theorem 3.2 by Ansel and Stricker [1994] since $\mathcal{P}_e \subseteq \mathcal{P}^T$. Furthermore, in the case $\bar{x}_u = -\infty$ the optimal claim is attainable by some self-financing strategy due to Theorem 2.3.9. However, in this case the corresponding value process is in general not bounded from below, and the expectation of X^* is not necessarily well defined under any $P \in \mathcal{P}$.

Proposition 2.3.8(ii) provides criteria for the property $P^* \sim Q^* \sim R$ required in the last part of Theorem 2.3.9. In this case the solution X^* is R-almost surely unique on the whole space. Moreover, Equations (2.44) and (2.47) show that the robust f-projection P^* of $\mathcal{Q}$ on $\mathcal{P}$ is a least favorable pricing measure for the optimal claim X^*. In the same manner, Equation (2.42) allows us to view Q^* as a least favorable measure for the utility evaluation of X^*. If Q^* minimizes the reverse f-divergence of P^* over the set $\mathcal{Q}$ simultaneously for all convex functions f, then Q^* is in fact a least favorable measure in the sense of Huber and Strassen [1973]; see Schied [2004] and Schied [2005b] for a more detailed discussion of the connection between robust utility maximization, risk measures, and the robust Neyman-Pearson lemma.

2.4 Applications and Examples

2.4.1 An Example

There are, in certain situations, means of determining the f-projection P_Q explicitly. This suggests the following method for finding the robust f-projection P^*: First calculate P_Q for each Q and then find the pair (P_Q, Q) with the smallest f-divergence. Here we want to give an example for this approach. P^* and Q^* will then give us the solution to the utility maximization problem.

In diffusion models for financial markets it is feasible to estimate the volatility of assets using historical data. However, estimations of the drift are much less reliable. Let us consider an example of a model in which the volatility and the structure of the drift are known, but there is uncertainty of the size of the drift.

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, R)$ be a two-dimensional Wiener space on which we are given two independent Brownian motions $B = (B_t)_{0\le t\le T}$ and $W = (W_t)_{0\le t\le T}$ with $B_0 = W_0 = 0$. We assume that $\mathcal{F} = \mathcal{F}_T$ and that $(\mathcal{F}_t)_{0\le t\le T}$ is the smallest right-continuous filtration that contains the filtration which is generated by the two Brownian motions. The discounted price process of an asset is modelled by

$$dS_t = S_t(\sigma_t dB_t + \mu_t dt) \quad (0 \le t \le T).$$

Finding equivalent local martingale measures for this model is equivalent to determining them for the model

$$d\tilde{S}_t := dB_t + \alpha_t dt \quad (0 \le t \le T)$$

with $\alpha = \mu/\sigma$. We assume that the process $\alpha = (\alpha_t)_{0\le t\le T}$ is B-integrable and adapted to the filtration $(\mathcal{F}_t^W)_{0\le t\le T}$ that is generated by the Brownian motion W.

For some interval $[b_1, b_2] \subseteq \mathbb{R}$, we define $\mathcal{Q}$ as the set of measures under which S has a drift of $b\mu$, or $\tilde{S}$ has a drift of $b\alpha$ for some $b \in [b_1, b_2]$, i.e.,

$$\mathcal{Q} := \left\{ Q_b : \frac{dQ_b}{dR} = \mathcal{E}\left((b-1)\int_0^T \alpha_s dB_s \right) \text{ for some } b \in [b_1, b_2] \right\},$$

where $\mathcal{E}$ is the Itô exponential

$$\mathcal{E}\left((b-1)\int_0^T \alpha_s dB_s \right) = \exp\left((b-1)\int_0^T \alpha_s dB_s - \frac{(b-1)^2}{2}\int_0^T \alpha_s^2 ds \right).$$

We let $B^{(b)}$ with $B_t^{(b)} := B_t - (b-1)\int_0^t \alpha_s ds$ be the corresponding Brownian motion under the measure Q_b.

We are going to consider the three standard utility functions $\log x$, $\frac{1}{\gamma}x^\gamma$, and $-\frac{1}{\alpha}e^{-\alpha x}$. Note that the υ_λ-projections and its reverse projections are independent of λ in these cases; see Example 2.1.8. So we may follow the simplified approach of first determining the υ-projections P_{Q_b} for any $Q_b \in \mathcal{Q}$. Then we will find the pair (P^*, Q^*) that minimizes the υ-divergence and hence also the υ_λ-divergence for any $\lambda > 0$. The constant λ^* will finally be calculated using the budget constraint.

Let us assume that in each case suitable integrability conditions hold.

(i) $u(x) = \log x$. For each $Q_b \in \mathcal{Q}$, the υ-projection P_{Q_b} has the density

$$\frac{dP_{Q_b}}{dQ_b} = \mathcal{E}\left(-\int_0^T b\alpha_s dB_s^{(b)} \right),$$

i.e., P_{Q_b} coincides with the minimal martingale measure (see Föllmer and Schweizer [1990]). This result was proven by Schweizer [1999] for general α. The f-divergence becomes

$$\begin{aligned}
v(P_{Q_b}|Q_b) &= \frac{b^2}{2} E_{Q_b}\left[\int_0^T \alpha_s^2 ds\right] \\
&= \frac{b^2}{2} E_R\left[E_R\left[\mathcal{E}\left((b-1)\int_0^T \alpha_s dB_s\right)\middle|\mathcal{F}_T^W\right]\int_0^T \alpha_s^2 ds\right] \\
&= \frac{b^2}{2} E_R\left[\int_0^T \alpha_s^2 ds\right].
\end{aligned}$$

The second equality holds due to the $\mathcal{F}_T^W$-measurability of $\int_0^T \alpha_s^2 ds$, and the last equality holds because $E_R[\mathcal{E}((b-1)\int_0^T \alpha_s^2 dB_s)|\mathcal{F}_T^W] = 1$ due to the independence of B from $\mathcal{F}_T^W$.

(ii) $u(x) = \frac{1}{\gamma}x^\gamma$. For each $Q_b \in \mathcal{Q}$, the v-projection P_{Q_b} has the density

$$\frac{dP_{Q_b}}{dQ_b} = C_b \exp\left(-\int_0^T b\alpha_s dS_s + \frac{1}{2(1-\gamma)} b^2 \int_0^T \alpha_s^2 ds\right)$$

with $C_b := E_{Q_b}\left[\exp\left(\beta\frac{b^2}{2}\int_0^T \alpha_s^2 ds\right)\right]^{-1}$ and $\beta := \gamma/(1-\gamma)$. This result is due to Grandits and Rheinländer [2002]. We have as in (i)

$$\begin{aligned}
v(P_{Q_b}|Q_b) &= \left(E_{Q_b}\left[\exp\left(\beta\frac{b^2}{2}\int_0^T \alpha_s^2 ds\right)\right]\right)^{(1+\beta)} \\
&= \left(E_R\left[\exp\left(\beta\frac{b^2}{2}\int_0^T \alpha_s^2 ds\right)\right]\right)^{(1+\beta)}.
\end{aligned}$$

(iii) $u(x) = -\frac{1}{\alpha}e^{-\alpha x}$. For each $Q_b \in \mathcal{Q}$, the v-projection P_{Q_b} has the density

$$\frac{dP_{Q_b}}{dQ_b} = C_b \exp\left(-\int_0^T b\alpha_s dS_s\right)$$

with $C_b := E_{Q_b}\left[\exp\left(-\frac{b^2}{2}\int_0^T \alpha_s^2 ds\right)\right]^{-1}$. This result was also shown by Grandits and Rheinländer [2002]. We have as in (i)

$$\begin{aligned}
v(P_{Q_b}|Q_b) &= -\log\left(E_{Q_b}\left[\exp\left(-\frac{b^2}{2}\int_0^T \alpha_s^2 ds\right)\right]\right) \\
&= -\log\left(E_R\left[\exp\left(-\frac{b^2}{2}\int_0^T \alpha_s^2 ds\right)\right]\right).
\end{aligned}$$

Now we see that for all three utility functions, the pair of measures that generates the smallest f-divergence is the one with $b^* = b_1$ if $b_1 > 0$, $b^* = b_2$ if $b_2 < 0$, and $b^* = 0$ if $b_1 < 0 < b_2$. In the last case, a martingale measure P^* is already contained in the set $\mathcal{Q}$ of subjective measures, and the pair (P^*, P^*), of course, minimizes the v-divergence. Hence in this model, $Q^* := Q_{b^*}$ is the measure that is closest to a martingale measure in the sense that it has the smallest drift in absolute value.

Now we can calculate the optimal contingent claim in all three cases.

(i) $u(x) = \log x$. Here we have $\lambda^* = 1/x_0$, and the optimal claim is given by

$$X^* = I\left(\lambda^* \frac{dP^*}{dQ^*}\right) = x_0 \exp\left(\int_0^T b^* \alpha_s dB_s^{(b^*)} + \frac{1}{2}\int_0^T (b^*)^2 \alpha_s^2 ds\right)$$
$$= x_0 \exp\left(\int_0^T \frac{b^*\alpha_s}{\sigma_s S_s} dS_s - \frac{1}{2}\int_0^T \left(\frac{b^*\alpha_s}{\sigma_s S_s}\right)^2 ds\right)$$
$$= x_0 + \int_0^T \xi_s dS_s,$$

where

$$\xi_s := \frac{b^*\alpha_s}{\sigma_s S_s} \exp\left(\int_0^T \frac{b^*\alpha_s}{\sigma_s S_s} dS_s - \frac{1}{2}\int_0^T \left(\frac{b^*\alpha_s}{\sigma_s S_s}\right)^2 ds\right).$$

(ii) $u(x) = \frac{1}{\gamma}x^\gamma$. In this case we have $\lambda^* = x_0^{\gamma-1} E_{Q^*}\left[\exp\left(\beta \frac{b^2}{2} \int_0^T \alpha_s^2 ds\right)\right]$, and the optimal claim is given by

$$X^* = x_0 \exp\left(\frac{1}{1-\gamma}\int_0^T b^*\alpha_s dS_s - \frac{1}{2(1-\gamma)^2}(b^*)^2 \int_0^T \alpha_s^2 ds\right)$$
$$= x_0 + \int_0^T \xi_s dS_s,$$

where

$$\xi_s := \frac{b^*\alpha_s}{1-\gamma} \exp\left(\frac{1}{1-\gamma}\int_0^s b^*\alpha_s dS_s - \frac{1}{2(1-\gamma)^2}(b^*)^2 \int_0^s \alpha_s^2 ds\right).$$

(iii) $u(x) = -\frac{1}{\alpha}e^{-\alpha x}$. Here $\lambda^* = -\alpha x_0 - \log C_{b^*}$ and

$$X^* = x_0 + \frac{1}{\alpha}\int_0^T b^*\alpha_s dS_s.$$

Hence for all three utility functions, the hedging strategy for the optimal claim consists of buying stocks if $b^*\alpha_s > 0$, selling stocks if $b^*\alpha_s < 0$, and if $b^* = 0$, we do not trade at all.

2.4.2 Example for the Dependence of the Worst Case Subjective Measure on the Utility Function

The previous example seems to suggest that the worst case subjective measure Q^* is independent of the utility function. This is the case if the set $\mathcal{Q}$ is a 2-alternating capacity as in Schied [2005b], Section 3.2. Let us here give a counter-example. The idea stems from the similar Example 1 in Huber and Strassen [1973], where it is shown that the set $\mathcal{Q} = \{Q_1, Q_2\}$ with Q_1 and Q_2 defined similarly to below is not a 2-alternating capacity.

Let Ω have four elements $\{A_1, A_2, A_3, A_4\}$, and let $\mathcal{F}$ be the power set of Ω. This can be interpreted as a two-period model for a financial market with four different states for the discounted stock price processes in the second period. For a probability measure Q on $(\Omega, \mathcal{F})$, let q_i be the probability of the event A_i $(i = 1, ..., 4)$ and write $Q = (q_1, q_2, q_3, q_4)$. Define the subjective measures $Q_1 := \left(\frac{5}{10}, \frac{2}{10}, \frac{2}{10}, \frac{1}{10}\right)$ and $Q_2 := \left(\frac{5}{10}, \frac{1}{10}, \frac{1}{10}, \frac{2}{10}\right)$, and let an equivalent martingale measure be given by $P := \left(\frac{11}{20}, \frac{2}{20}, \frac{4}{20}, \frac{3}{20}\right)$. This could, for instance, be the unique equivalent martingale measure in a market model with three stocks, each starting at a price of 1, and combinations of the three discounted stock prices after one period of $A_1 := (1.1;\ 1.2;\ 1.5)$, $A_2 := (1.1;\ 0.4;\ 7/18)$, $A_3 := (57/70;\ 1.2;\ 7/18)$, and $A_4 := (57/70;\ 0.4;\ 7/18)$. We define $\mathcal{Q}$ as the convex hull of the set $\{Q_1, Q_2\}$. If v is strictly convex, then due to the strict convexity of the v-divergence it is obvious that the reverse v-projection of P is either Q_1 or Q_2.

Let $u^{(1)}(x) := 2\sqrt{x}$. Then the convex conjugate is given by $v^{(1)}(x) := \frac{1}{x}$, and

$$v^{(1)}(P|Q) = \frac{q_1^2}{p_1} + \frac{q_2^2}{p_2} + \frac{q_3^2}{p_3} + \frac{q_4^2}{p_4}.$$

Hence,

$$v^{(1)}(P|Q_1) = \frac{25}{10} \cdot \frac{2}{11} + \frac{4}{10} + \frac{4}{10} \cdot \frac{2}{4} + \frac{1}{10} \cdot \frac{2}{3} = \frac{37}{33} = \frac{740}{660},$$

and

$$v^{(1)}(P|Q_2) = \frac{36}{10} \cdot \frac{2}{11} + \frac{1}{10} + \frac{1}{10} \cdot \frac{2}{4} + \frac{4}{10} \cdot \frac{2}{3} = \frac{707}{660}.$$

Now let $u^{(2)}(x) := 1 - e^{-x}$. Then the convex conjugate is given by $v^{(2)}(x) := x \log x$, and

$$v^{(2)}(P|Q) = p_1 \log \frac{p_1}{q_1} + p_2 \log \frac{p_2}{q_2} + p_3 \log \frac{p_3}{q_3} + p_4 \log \frac{p_4}{q_4}.$$

Hence,

$$v^{(2)}(P|Q_1) = \frac{11}{20}\log\frac{11}{2\cdot 5} + \frac{1}{10}\log\frac{1}{2} + \frac{2}{10}\log 1 + \frac{3}{20}\log\frac{3}{2} \approx 0.019,$$

and

$$v^{(2)}(P|Q_2) = \frac{11}{20}\log\frac{11}{2\cdot 6} + \frac{1}{10}\log 1 + \frac{2}{10}\log 2 + \frac{3}{20}\log\frac{3}{4} \approx 0.021.$$

Since the reverse v-projections are independent of λ is theses cases, Q_2 is the reverse $v_\lambda^{(1)}$-projection, but Q_1 is the reverse $v_\lambda^{(2)}$-projection for any $\lambda > 0$.

2.4.3 Expenditure Minimization

A problem that is closely related to the one of utility maximization is the minimization of expenditures given the agent has a minimum level y_0 of robust expected utility, i.e,

$$\begin{aligned} &\text{Minimize } \sup_{P\in\mathcal{P}'} E_P[Y] \text{ over contingent claims } Y \geq \bar{x}_u \\ &\qquad\text{that satisfy } \inf_{Q\in\mathcal{Q}} E_Q[u(Y)] \geq y_0. \end{aligned} \tag{2.53}$$

The constraint can also be interpreted as a maximum level of robust expected loss by defining a convex loss function as $\ell(x) := -u(-x)$; see also Chapter 3. Applying the methods that were introduced in this chapter it is now easy to solve this problem. Instead of Lemma 2.3.1, we now need

Lemma 2.4.1. *Assume that $v(\mathcal{P}'|\mathcal{Q}) < \infty$. For $y_0 \in (u(\bar{x}_u), u(\infty))$, the function*

$$\hat{H}(\lambda) := \lambda v_{1/\lambda}(\mathcal{P}'|\mathcal{Q}) - \lambda y_0 = \inf_{P\in\mathcal{P}'}\inf_{Q\in\mathcal{Q}} E_R\left[v\left(\frac{dP}{dR}, \lambda\frac{dQ}{dR}\right)\right] - \lambda y_0$$

is convex and achieves its infimum in some finite value $\hat{\lambda} > 0$.

Proof. Convexity is shown in exactly the same way as in Lemma 2.3.1. It remains to show that $0 < \hat{\lambda} < \infty$, and this proof also follows the lines of the proof of Lemma 2.3.1: From Jensen's inequality it follows that

$$\lim_{\lambda\to\infty}\hat{H}(\lambda) \geq \lim_{\lambda\to\infty}\left(\lambda v\left(\frac{1}{\lambda}\right) - \lambda y_0\right) = \lim_{\lambda\to\infty}\lambda\left(v\left(\frac{1}{\lambda}\right) - y_0\right) = \infty,$$

where the last equality follows from $v(0) = u(\infty)$ shown Lemma 2.1.6 and the fact that $y_0 < u(\infty)$. In order to show that $\hat{\lambda} > 0$, observe that by

Jensen's inequality $\lim_{\lambda\to 0}\hat{H}(\lambda) \geq \lim_{\lambda\to 0}\lambda v\left(1/\lambda\right) = v(\infty)/\infty$. If this is ∞, it is obvious that $\hat{\lambda} > 0$. If $v(\infty)/\infty = 0$, assume that $\hat{\lambda} = 0$. We can choose $P \in \mathcal{P}'$ and $Q \in \mathcal{Q}$ such that $v(P|Q) < \infty$, and then $v_{1/\lambda}(P|Q) < \infty$ for all $\lambda > 0$ due to Lemma 2.2.1. Hence for any $\lambda > 0$,

$$\begin{aligned}
0 \leq \hat{H}(0) &\leq \hat{H}(\lambda) \\
&\leq \lambda v_{1/\lambda}(P|Q) - \lambda y_0 \\
&\leq \lambda\left(E_Q\left[u\left(I\left(\frac{1}{\lambda}\frac{dP}{dQ}\right)\right)\right] - y_0\right),
\end{aligned}$$

where the last inequality follows as in the proof of Lemma 2.3.1 from the convexity of v and the fact that $\partial v(x,y)/\partial y = u(I(x/y))$. Due to Lemma 2.2.1 and the fact that $y_0 > u(\bar{x}_u)$, we can choose $\lambda > 0$ such that the expression in the bracket is negative which leads to a contradiction. □

Let us now state the solution to Problem (2.53) and show its relation to the robust utility maximization problem.

Proposition 2.4.2. *Let the assumptions of Theorems 2.3.9 or 2.3.10 hold depending on whether $\bar{x}_u = -\infty$ or $\bar{x}_u = 0$. Let $\hat{P} \in \mathcal{P}'$ and $\hat{Q} \in \mathcal{Q}$ be the minimizer of $v_{1/\hat{\lambda}}(P|Q)$ over $\mathcal{P}'$ and $\mathcal{Q}$.*

(i) The solution to the expenditure minimization problem (2.53) *is given by*

$$Y^* := I\left(\frac{1}{\hat{\lambda}}\frac{d\hat{P}}{d\hat{Q}}\right), \tag{2.54}$$

and the minimal cost equals

$$E_{\hat{P}}[Y^*] := \hat{\lambda} v_{1/\hat{\lambda}}(\mathcal{P}'|\mathcal{Q}) - \hat{\lambda} y_0.$$

(ii) Let $X^* := I\left(\lambda^*\frac{dP^*}{dQ^*}\right)$ be the solution to the utility maximization problem (2.40) *as in Theorems 2.3.9 and 2.3.10, respectively. If $y_0 = E_{Q^*}\left[u\left(X^*\right)\right]$, then X^* is also the solution to the expenditure minimization problem* (2.53).

(iii) Let $Y^* := I\left(\frac{1}{\hat{\lambda}}\frac{d\hat{P}}{d\hat{Q}}\right)$ be the solution to the expenditure minimization problem (2.53). *If $x_0 = E_{P^*}\left[Y^*\right]$, then Y^* is also the solution to the utility maximization problem* (2.40) *as in Theorems 2.3.9 and 2.3.10, respectively.*

Proof. (i) The existence of $\hat{P}$ and $\hat{Q}$ follows from Theorems 1.1.2 and 1.2.8. Take $Y \geq \bar{x}_u$ that satisfies the constraint $\inf_{Q\in\mathcal{Q}} E_Q[u(Y)] \geq y_0$. Then with the same notation as in Theorems 2.3.9 and 2.3.10 we have

$$\begin{aligned}\sup_{P\in\mathcal{P}'_0} E_P[Y] \geq E_{\hat{P}}[Y] &\geq E_{\hat{P}}[Y] + \hat{\lambda}\left(y_0 - E_{\hat{Q}}[u(Y)]\right)\\ &= -E_R\left[\hat{\lambda}\frac{d\hat{Q}}{dR}u(Y) - \frac{d\hat{P}}{dR}Y\right] + \hat{\lambda}y_0\\ &\geq -E_R\left[v\left(\frac{d\hat{P}}{dR}, \hat{\lambda}\frac{d\hat{Q}}{dR}\right)\right] + \hat{\lambda}y_0\\ &= E_{\hat{P}}\left[I\left(\frac{1}{\hat{\lambda}}\frac{d\hat{P}}{d\hat{Q}}\right)\right].\end{aligned}$$

The last step follows from the fact that $\hat{\lambda}$ is a minimizer of $\lambda v_{1/\lambda}(\hat{P}|\hat{Q}) - \lambda y_0$ and the fact that $\partial v(x,y)/\partial y = u(I(x/y))$ as in Lemma 2.2.2: We have $y_0 = E_{\hat{Q}}\left[u\left(I\left(\frac{1}{\hat{\lambda}}\frac{d\hat{P}}{d\hat{Q}}\right)\right)\right]$. Furthermore, it follows from Lemma 2.3.8 that $\hat{P}$ and $\hat{Q}$ are worst case measures, i.e., $E_{\hat{P}}[Y^*] = \sup_{P\in\mathcal{P}'_0} E_P[Y^*]$ and $E_{\hat{Q}}[u(Y^*)] = \inf_{Q\in\mathcal{Q}} E_Q[u(Y)]$, which shows that Y^* satisfies the constraint and is a solution to Problem (2.53). The second part of (i) now follows from the above inequalities.

(ii) Define $G(\lambda) := v_\lambda(\mathcal{P}'|\mathcal{Q})$ and $\hat{G}(\lambda) := \lambda v_{1/\lambda}(\mathcal{P}'|\mathcal{Q})$. By (2.14) and Theorems 2.3.9 and 2.3.10 $y_0 = E_{Q^*}[u(X^*)] = v_{\lambda^*}(P^*|Q^*) + \lambda^* E_{P^*}[X^*] = G(\lambda^*) + \lambda^* x_0$ because X^* satisfies the budget constraint with equality. Hence $x_0 = (y_0 - G(\lambda^*))/\lambda^*$, and λ^* is a minimizer of $H(\lambda) := G(\lambda) + \lambda x_0$ if and only if

$$y_0 = G(\lambda^*) + \lambda^* x_0 \leq G(\lambda) + \lambda x_0 = G(\lambda) + \frac{\lambda}{\lambda^*}(y_0 - G(\lambda^*))$$

for all $\lambda > 0$. But since $\hat{G}(\lambda) = \lambda G(1/\lambda)$, this implies

$$\hat{G}\left(\frac{1}{\lambda^*}\right) - \frac{y_0}{\lambda^*} \leq \hat{G}\left(\frac{1}{\lambda}\right) - \frac{y_0}{\lambda}$$

for all $\lambda > 0$. Hence $1/\lambda^*$ minimizes $\hat{H}(\lambda) = \hat{G}(\lambda) - \lambda y_0$, and by (i) X^* is also a solution to the expenditure minimization problem.

(iii) It follows from (i) that $x_0 = E_{P^*}[Y^*] = -\hat{G}(\hat{\lambda}) + \hat{\lambda}y_0$. Hence $y_0 = (x_0 + \hat{G}(\hat{\lambda}))/\hat{\lambda}$, and $\hat{\lambda}$ is a minimizer of $\hat{H}(\lambda) = \hat{G}(\lambda) - \lambda y_0$ if and only if

$$-x_0 = \hat{G}(\hat{\lambda}) - \hat{\lambda}y_0 \leq \hat{G}(\lambda) - \lambda y_0 = \hat{G}(\lambda) - \frac{\lambda}{\hat{\lambda}}(x_0 + \hat{G}(\hat{\lambda}))$$

for all $\lambda > 0$. But since $\hat{G}(\lambda) = \lambda G(1/\lambda)$, this implies

$$G\left(\frac{1}{\hat{\lambda}}\right) + \frac{x_0}{\hat{\lambda}} \leq G\left(\frac{1}{\lambda}\right) + \frac{x_0}{\lambda}$$

for all $\lambda > 0$. Hence $1/\hat{\lambda}$ minimizes $H(\lambda) = G(\lambda) + \lambda x_0$, and by Theorems 2.3.9 and 2.3.10 Y^* is also a solution to the utility maximization problem. $\square$

Remark 2.4.3. *(i) shows that the solution to the expenditure minimization problem is again of the form* (2.54) *as in the utility maximization problem. This leads to (ii) and (iii), where the relation of the solutions to these two problems is shown. This is a well known fact in the utility maximization theory of deterministic payoffs; see, for instance, Mas-Colell et al. [1995], Proposition 3.E.1.*

2.5 Conclusion

We characterize the solution to the robust utility maximization problem (2.40) in an incomplete market model and prove its existence. This is done using a general martingale or duality approach. We first solve the classical problem (2.22) under the budget constraint in terms of a single martingale measure. Then we apply the existence results from Sections 1.1 and 1.2 from the previous chapter to solve the dual problem. To this end, we have to distinguish between utility functions that are finite on the whole real line and those which are only defined for positive values. This leads to a different budget constraint in each case. We first show that the minimizing measures from the dual problem can be interpreted as worst case measures and then use this characterization in order to solve the general utility maximization problem.

Furthermore, we give an example of a stock price process which is driven by a generalized geometric Brownian motion with uncertainty of the drift. Here it turns out that the subjective worst case measure is the same for all three considered utility function: It is the one with the smallest possible drift. We illustrate by another example that this is not always the case. Finally, we solve the problem of determining a claim which minimizes the expenditures under the constraint of a minimum robust expected utility.

Chapter 3

Utility Maximization Under a Shortfall Risk Constraint

For financial institutions, the measurement and management of downside risk is a key issue. Value at Risk (VaR) has emerged as the industry standard for risk measurement but shows serious deficiencies as a measure of downside risk. It penalizes diversification in many situations and does not take into account the size of very large losses exceeding the value at risk. These problems motivated intense research on alternative risk measures whose foundation was provided by Artzner et al. [1999].

A good *risk measure* needs to have several virtues. First, it should measure risk on a monetary scale: the notion of risk entails the amount of capital we need to set aside in order to make a position acceptable from a risk management perspective. Second, a risk measure should penalize concentrations and encourage diversification. Third, a risk measure should be sensitive to the size of losses. Taking a more practical perspective, a risk measure should also be easily estimated from simulations of profit and loss distributions. Many characterization theorems for alternative families of risk measures are now available. An summary of recent results can be found in the book by Föllmer and Schied [2004].

While these results are an important first step towards better risk management, an analysis of the economic implications of different approaches to risk measurement is indispensable. In this chapter we investigate the agent's optimal payoff profile under a joint budget and risk measure constraint. We define the risk constraint in terms of *utility-based shortfall risk* (UBSR). In order to analyze the impact of the downside risk constraint, we discuss two examples and compare the solutions to both utility maximization without risk constraint and under a VaR constraint. While the risk measure VaR limits the probability of a loss, it actually leads to large losses in these events. This deficiency is not shared by the family of utility-based shortfall risk measures. In fact, UBSR measures possess all the virtues which we discussed above. For a detailed description of their properties, we refer to Föllmer and Schied [2004], Weber [2005], Dunkel and Weber [2005], and Giesecke et al. [2005].

This chapter is organized as follows. In Section 3.1 we present the constrained maximization problems, first the simplified one in a "complete market" situation without model uncertainty, and then the general one in an incomplete market under model uncertainty. Section 3.2 is devoted to the solution of the simplified problem. In Proposition 3.2.2 we first solve an auxiliary problem which consists of the minimization of the expected loss. In Theorem 3.2.3 we show that a solution to the constrained utility maximization problem exists, and we characterize this solution by four possible situations: There is no contingent claim that satisfies both constraints; the solution to the utility maximization problem equals the loss-minimizing claim; the risk constraint is not binding, and the solution coincides with the solution of the problem without risk constraint from the previous chapter; the risk constraint is binding, and the solution is given is terms of a deterministic function of the densities of certain measures. The most challenging part is the proof of the existence of a pair of Lagrange multipliers that guarantees that both constraints are satisfied and hence a solution exists. This result is given in Lemma 3.2.4, and the proof can be found in Section 3.2.2. In order to keep the presentation as clear as possible, we separately give some duality results in Section 3.2.1. In Section 3.3 we then consider the general robust problem in an incomplete market. We also first solve a (robust) loss minimization problem in Proposition 3.3.6. Similarly to the procedure in Chapter 2 we then give a characterization of certain minimizing measures as worst case measures in Proposition 3.3.12, which allows us to finally solve the robust utility maximization problem in an incomplete market under both a budget and a risk constraint. This solution is given in Theorem 3.3.13. In Section 3.4 we discuss specific examples of price processes, namely geometric Brownian motion and a geometric Poisson process and compare the solutions to the optimal claims under a VaR constraint and without a risk constraint.

3.1 The Constrained Maximization Problem

Let us consider the market model $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\le t\le T}, R)$ from Section 2.1. Let $\mathcal{Q}_0$ be a set of subjective probability measures on $(\Omega, \mathcal{F})$, which are now assumed to be equivalent to the reference measure R. We want to maximize the robust utility functional

$$U(X) = \inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X)], \tag{3.1}$$

where the utility function u satisfies all assumptions from Section 2.1. Recall the definition of $\bar{x}_u := \inf\{x : u(x) > -\infty\}$ as the left boundary of the domain of the utility function. We will investigate this maximization problem in presence of both a budget and a risk constraint.

As in Section 2.2, we will start with the classical utility maximization problem in a "complete market" situation and work with the constraint

$$E_P[X] \leq x_0. \tag{3.2}$$

This can be interpreted as the budget constraint in a complete market, where P is the unique equivalent martingale measure. However, we only assume that $P \in \mathcal{P}'$, i.e., P is either some absolutely continuous martingale measure $P \in \mathcal{P}$ if $\bar{x}_u = -\infty$, or P is the projection $P^T \in \mathcal{P}^T$ of an extended martingale measure $\bar{P} \in \bar{\mathcal{P}}$ on $(\Omega, \mathcal{F}_T)$ if $\bar{x}_u = 0$. Extended martingale measures were introduced in Section 1.2.

Then we will consider the utility maximization problem in an incomplete market under model uncertainty in the case where the utility function is only defined on the positive halfline, i.e., $\bar{x}_u = 0$. As in Section 2.3.1, a contingent claim X is affordable if there exists a self-financing strategy whose value process dominates X at the final time T. Since we will suppose $X \geq 0$, the optional decomposition theorem by Kramkov [1996] and its generalization by Föllmer and Kabanov [1998] imply that the budget constraint is given by

$$\sup_{P \in \mathcal{P}_e} E_P[X] \leq x_0; \tag{3.3}$$

see Lemma 2.3.4. This Lemma also shows that (3.3) is equivalent to

$$\sup_{P \in \mathcal{P}^T} E_P[X] \leq x_0, \tag{3.4}$$

and as in Chapter 2 we will work with this constraint. Note that we again drop the superscript T and just write P for a projection of an extended martingale measure $\bar{P} \in \bar{\mathcal{P}}$ on $(\Omega, \mathcal{F}_T)$.

In the robust case we need the following assumption on the set $\mathcal{Q}_0$ of subjective measures.

Assumption 3.1.1. *We assume that all measures in the convex set $\mathcal{Q}_0$ are equivalent to the reference measure R and that the set of densities*

$$\mathcal{K}_{\mathcal{Q}_0} := \left\{ \frac{dQ_0}{dR} : \ Q_0 \in \mathcal{Q}_0 \right\} \tag{3.5}$$

is weakly compact in $L^1(R)$.

3.1.1 The Risk Constraint

The risk of a financial position can be quantified by appropriate risk measures. We let $\mathcal{D}$ be some vector space of random variables.

Definition 3.1.2. *A mapping $\rho : \mathcal{D} \to \mathbb{R}$ is called a* risk measure *(on $\mathcal{D}$) if it satisfies the following conditions for all $X_1, X_2 \in \mathcal{D}$:*

(i) Inverse Monotonicity: *If $X_1 \leq X_2$, then $\rho(X_1) \geq \rho(X_2)$.*

(ii) Translation Invariance*: If $m \in \mathbb{R}$, then $\rho(X + m) = \rho(X) - m$.*

Monotonicity refers to the property that risk decreases if the payoff profile is increased. Translation invariance formalizes that risk is measured on a monetary scale: if a monetary amount $m \in \mathbb{R}$ is added to a position X, then the risk of X is reduced by m.

Value at risk (VaR in the following) is a risk measure according to the above definition, but it does in general not encourage diversification of positions since it is not a convex risk measure if $L^\infty \subseteq \mathcal{D}$. A risk measure ρ is *convex* (on $\mathcal{D}$) if it satisfies the following conditions for all $X_1, X_2 \in \mathcal{D}$:

(iii) *Convexity:* $\rho(\alpha X_1 + (1-\alpha)X_2) \leq \alpha\rho(X_1) + (1-\alpha)\rho(X_2)$ for all $\alpha \in (0, 1)$.

Here we focus on a particular example of a convex risk measure, namely *utility-based shortfall risk*. Utility-based shortfall risk is most easily defined as a *capital requirement*, i.e., the smallest monetary amount that has to be added to a position to make it acceptable.[1] We will now give the definition of utility-based shortfall risk.

Let $\ell : \mathbb{R} \to [0, \infty]$ be a loss function, i.e., an increasing function that is not constant. The level x_1 shall be a point in the interior of the range of ℓ. Let Q_1 be a fixed subjective probability measure equivalent to R, which we will use for the purpose of risk management. For example, in our model one could suppose that both Q_1 and Q_0 signify the empirical real world measure. Or Q_0 could be the subjective measure that the agent chooses, and Q_1 the subjective measure of a supervising agency. The space of financial positions $\mathcal{D}$ is chosen in such a way that for $X \in \mathcal{D}$ the integral $\int \ell(-X)dQ_1$ is well defined.

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[1] Note that every static risk measure can be defined as a capital requirement. To be more precise, if ρ is a risk measure, then $\mathcal{A} = \{X \in \mathcal{D} : \rho(X) \leq 0\}$ defines its acceptance set, i.e., the set of positions with non-positive risk. ρ is then recovered as

$$\rho(X) = \inf\{m \in \mathbb{R} : X + m \in \mathcal{A}\};$$

see Föllmer and Schied [2004], Chapter 4.

Define an acceptance set

$$\mathcal{A}_{Q_1} = \{X \in \mathcal{D}: \ E_{Q_1}[\ell(-X)] \leq x_1\}. \tag{3.6}$$

A financial position is thus acceptable if the expected value of $\ell(-X)$ under the subjective probability measure Q_1, i.e., the expected loss $E_{Q_1}[\ell(-X)]$, is not more than x_1.[2]

The acceptance set $\mathcal{A}_{Q_1}$ induces the risk measure *utility-based shortfall risk* (*UBSR* in the following) ρ_{Q_1} as the associated capital requirement

$$\rho_{Q_1}(X) = \inf\{m \in \mathbb{R}: \ X + m \in \mathcal{A}_{Q_1}\}. \tag{3.7}$$

Utility-based shortfall risk is convex and does therefore encourage diversification. Examples of loss functions ℓ include exponentials $\exp(\alpha x)$, $\alpha > 0$, which lead to the so-called *entropic risk measures*, for which a simple explicit formula is available; see Föllmer and Schied [2004], Example 4.105. Alternatively, one-sided loss functions can be used to measure downside risk only. These risk measures look at losses only and do not consider trade-offs between gains and losses. Examples include $x^{\alpha} \cdot 1_{(0,\infty)}(x)$, $\alpha > 1$, or exponentials $\exp(\alpha x) \cdot 1_{(0,\infty)}(x)$, $\alpha > 0$.

Our aim is to solve the utility maximization problem under a joint budget and risk measure constraint. If there is no model uncertainty, the shortfall risk constraint (*UBSR constraint* in the following) shall be given by

$$\rho_{Q_1}(X) \leq 0. \tag{3.8}$$

A financial position X which satisfies (3.8) is acceptable from the point of view of the risk measure ρ. This is equivalent to

$$E_{Q_1}[\ell(-X)] \leq x_1. \tag{3.9}$$

In the case where the agent faces model uncertainty, let us consider a second set $\mathcal{Q}_1$ of subjective measures which are equivalent to the reference measure R. This set may, of course, coincide with $\mathcal{Q}_0$, the set of subjective measures for the utility evaluation in (3.1). The *robust UBSR constraint* is given by

$$\sup_{Q_1 \in \mathcal{Q}_1} \rho_{Q_1}(X) \leq 0. \tag{3.10}$$

[2]We have defined acceptability in terms of a loss function ℓ. Alternatively, we could define $u_\ell(x) = -\ell(-x)$ and interpret u_ℓ as a *utility function.* $U_\ell(X) = E_{Q_1}[u_\ell(X)]$ defines in this case a utility functional. X is thus acceptable if its utility is at least $-x_1$. This explains why the risk measure is called *utility-based.*

That is, any financial position must be acceptable from the point of view of *all* risk measures ρ_{Q_1} $(Q_1 \in \mathcal{Q}_1)$. This is equivalent to

$$\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X)] \leq x_1. \tag{3.11}$$

As for the set $\mathcal{Q}_0$, we need

Assumption 3.1.3. *We assume that all measures in the convex set $\mathcal{Q}_1$ are equivalent to the reference measure R and that the set of densities*

$$\mathcal{K}_{Q_1} := \left\{ \frac{dQ_1}{dR} : \; Q_1 \in \mathcal{Q}_1 \right\} \tag{3.12}$$

is weakly compact in $L^1(R)$.

We require the loss function ℓ to satisfy the following technical conditions. We assume that ℓ is strictly convex, strictly increasing, and continuously differentiable on the interval $(-\bar{x}_\ell, \infty)$ for some $\bar{x}_\ell \in \mathbb{R} \cup \{\infty\}$, that $\ell(x) = 0$ for $x \leq -\bar{x}_\ell$, and that ℓ is continuous on the whole real line, and $\lim_{x \to -\infty} \ell(x) = 0$ and $\lim_{x \to -\infty} \ell'(x) = 0$ if $\bar{x}_\ell = \infty$. As for the utility function in Section 2.1, we assume that ℓ has regular asymptotic elasticity, i.e.,

$$\begin{aligned} \limsup_{x \to \infty} \frac{x\ell'(x)}{\ell(x)} &> 1 \quad \text{if } \bar{x}_u = -\infty, \qquad \text{and} \\ \liminf_{x \to -\infty} \frac{x\ell'(x)}{\ell(x)} &< 1 \quad \text{if } \bar{x}_\ell = \infty. \end{aligned} \tag{3.13}$$

Note that, when we compare (3.13) to Assumption (2.9), the signs switch since we in fact require that the corresponding utility function $u_\ell := -\ell(-x)$ has reasonable asymptotic elasticity.

3.1.2 The Non-Robust Problem in a "Complete Market" Setting

Let us fix some martingale measure $P \in \mathcal{P}'$, i.e., an absolutely continuous martingale measure $P \in \mathcal{P}$ if $\bar{x}_u = -\infty$, and the projection P^T of an extended martingale measure $\bar{P} \in \bar{\mathcal{P}}$ on $(\Omega, \mathcal{F}_T)$ if $\bar{x}_u = 0$. Furthermore, fix a subjective measure $Q_0 \in \mathcal{Q}_0$ for the utility evaluation, and a subjective measure $Q_1 \in \mathcal{Q}_1$ for the risk constraint. Note that in this chapter we assume that $Q_1 \sim Q_0 \sim R$. We denote the set of terminal financial positions with well defined utility by

$$\mathcal{I}_{P,Q_0} = \left\{ X \geq \bar{x}_u : \;\; X \in L^1(P) \text{ and } u(X)^- \in L^1(Q_0) \right\}. \tag{3.14}$$

Fix an initial endowment $x_0 > \bar{x}_u$ and let $x_1 > 0$ be a risk limit according to the definition of UBSR. We are now able to formulate the optimization problem under a joint budget and UBSR constraint:

$$\begin{aligned} &\text{Maximize } E_{Q_0}[u(X)] \text{ over all } X \in \mathcal{I}_{P,Q_0} \\ &\text{that satisfy } E_{Q_1}[\ell(-X)] \leq x_1 \text{ and } E_P[X] \leq x_0. \end{aligned} \tag{3.15}$$

The set of all financial positions in $\mathcal{I}_{P,Q_0}$ that satisfy the two constraints is denoted by $\mathcal{X}_{P,Q_1,Q_0}(x_0, x_1)$, i.e.,

$$\mathcal{X}_{P,Q_1,Q_0}(x_0, x_1) := \{X \in \mathcal{I}_{P,Q_0} : \ E_{Q_1}[\ell(-X)] \leq x_1 \text{ and } E_P[X] \leq x_0\}. \tag{3.16}$$

If $\bar{x}_u = 0$, then we may and will always assume without loss of generality that $\bar{x}_\ell \in (\bar{x}_u, \infty]$. Since any contingent claim with utility larger than $-\infty$ does not take any values below $\bar{x}_u$ with positive probability, any loss constraint with $\bar{x}_\ell \leq \bar{x}_u$ is trivially satisfied, and we are back in the classical case without any risk constraint.

3.1.3 The Robust Problem in an Incomplete Market Model

In the incomplete market case we assume that $\bar{x}_u = 0$. Let us denote the set of terminal financial positions with well defined utility by

$$\mathcal{I} = \left\{X \geq 0 : \ \ X \in L^1(P)\ \forall\, P \in \mathcal{P}^T \text{ and } u(X)^- \in L^1(Q_0)\ \forall\, Q_0 \in \mathcal{Q}_0\right\}. \tag{3.17}$$

For x_0, $x_1 > 0$, we then want to solve the optimization problem under a joint budget and UBSR constraint:

$$\begin{aligned} &\text{Maximize } \inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X)] \text{ over all } X \in \mathcal{I} \\ &\text{that satisfy } \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X)] \leq x_1 \text{ and } \sup_{P \in \mathcal{P}^T} E_P[X] \leq x_0. \end{aligned} \tag{3.18}$$

The set of all financial positions in $\mathcal{I}$ that satisfy the two constraints is denoted by $\mathcal{X}$, i.e.,

$$\mathcal{X}(x_0, x_1) := \{X \in \mathcal{I} : \ \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X)] \leq x_1 \text{ and } \sup_{P \in \mathcal{P}^T} E_P[X] \leq x_0\}. \tag{3.19}$$

As in Chapter 2, we will first solve the simplified problem (3.15) without model uncertainty and then use this result to tackle the general problem (3.18).

3.2 The Solution to the Non-Robust Problem in a "Complete Market" Setting

We will show that under suitable integrability assumptions the unique solution to the constrained maximization problem (3.15) can be written in the form

$$X_{P,Q_1,Q_0} = x^* \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right),$$

where $x^* : [0, \infty) \times (0, \infty) \to (\bar{x}_u, \infty)$ is a continuous deterministic function, and λ_1^*, λ_2^* are suitable real parameters. x^* is obtained as the solution of a family of deterministic maximization problems. The details and proofs for the following claims can be found in Lemma 3.2.12 in Section 3.2.2 below.

We define a family of functions g_{y_1,y_2} with $y_1, y_2 \geq 0$ by

$$g_{y_1,y_2}(x) := u(x) - y_1 \ell(-x) - y_2 x.$$

For each pair $y_1 \geq 0, y_2 > 0$, the maximizer of g_{y_1,y_2} is unique and equals

$$x^*(y_1, y_2) \quad := \quad \begin{cases} J(y_1, y_2) & \text{if } y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell+), \\ \bar{x}_\ell & \text{if } u'(\bar{x}_\ell) \leq y_2 \leq u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell+), \\ I(y_2) & \text{if } y_2 < u'(\bar{x}_\ell); \end{cases} \tag{3.20}$$

see Section 3.2.2. Here, $J(y_1, y_2)$ denotes the unique solution to the equation $u'(x) + y_1 \ell'(-x) = y_2$ for the case that $y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell+)$, and $I := (u')^{-1}$. Note that $x^*(0, y_2) = I(y_2) = J(0, y_2)$.

The derivation of the solution to (3.15) requires as prerequisite the solution of a related problem. We need to determine a financial position $Y_{P,Q_1} \geq \bar{x}_u$ that minimizes the expected loss under the budget constraint (3.2). That is, we have to solve the problem

$$\begin{gathered} \text{Minimize } E_{Q_1}[\ell(-Y)] \text{ over all financial positions } Y \geq \bar{x}_u \\ \text{with } Y \in L^1(P) \text{ and } E_P[Y] \leq x_0. \end{gathered} \tag{3.21}$$

We will see that the solution to this problem is of the form

$$Y_{P,Q_1} = -L \left(c_{P,Q_1} \frac{dP}{dQ_1} \right).$$

Here $L : \mathbb{R} \to [-\bar{x}_\ell, -\bar{x}_u]$ is defined as the generalized inverse of the derivative of the loss function ℓ, i.e.,

$$L(y) := \begin{cases} -\bar{x}_u & \text{if } y \geq \ell'(-\bar{x}_u), \\ (\ell')^{-1}(y) & \text{if } \ell'(-\bar{x}_\ell+) < y < \ell'(-\bar{x}_u), \\ -\bar{x}_\ell & \text{if } y \leq \ell'(-\bar{x}_\ell+). \end{cases} \tag{3.22}$$

L is a continuous function which is strictly increasing on $[\ell'(-\bar{x}_\ell+), \ell'(-\bar{x}_u)]$. Recall that $\bar{x}_\ell$ might assume the value ∞, and $\bar{x}_u$ might be $-\infty$. Properties of the functions x^* and L are collected in Section 3.2.2.

In order to guarantee the existence of our solution to the optimization problem (3.15), we make the following technical assumptions.

Assumption 3.2.1. *Let the functions x^* and L be defined as in* (3.20) *and* (3.22)*. We impose the following integrability assumptions for all $\lambda_1 \geq 0, \lambda_2 > 0$, and $c > 0$:*

(a) $x^*\left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0}\right) \in L^1(P)$,

(b) $\ell\left(-x^*\left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0}\right)\right) \in L^1(Q_1)$,

(c) $u\left(x^*\left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0}\right)\right) \in L^1(Q_0)$,

(d) $L\left(c\frac{dP}{dQ_1}\right) \in L^1(P)$,

(e) $\ell\left(L\left(c\frac{dP}{dQ_1}\right)\right) \in L^1(Q_1)$.

Assumption 3.2.1 extends the standard integrability conditions from our Lemma 2.2.1 to the case of utility maximization under a joint budget and UBSR constraint. Assumptions (a)-(c) guarantee that price, expected loss and utility of the solution are well defined. Assumptions (d) and (e) impose integrability of the solution to the loss minimization problem (3.21), which is an intermediate step in the analysis of problem (3.15). In contrast to the utility maximization problem without risk constraint, the existence of a solution to (3.15) is not immediate from Assumption 3.2.1, but requires a sophisticated analysis of the constraints, see Lemma 3.2.4 and Section 3.2.2 below. In Section 3.2.1 we will show that Assumption 3.2.1 is equivalent to the finiteness of two convex functionals, similarly to the results from Lemma 2.2.1.

Let us now state the solution to the loss minimization problem (3.21).

Proposition 3.2.2. *Suppose that Assumptions 3.2.1(d)&(e) hold and let $x_0 \in (\bar{x}_u, \bar{x}_\ell)$. Then the equation*

$$x_0 = -E_P\left[L\left(c\frac{dP}{dQ_1}\right)\right] \tag{3.23}$$

has a solution $c_{P,Q_1} > 0$. A solution to Problem (3.21) *is given by*

$$Y_{P,Q_1} := -L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right). \tag{3.24}$$

On the set $\{dP/dR > 0\}$, the loss minimizing contingent claim is R-almost surely unique, i.e., $Y_{P,Q_1} \cdot 1_{\{dP/dR>0\}} = \tilde{Y} \cdot 1_{\{dP/dR>0\}}$ R-almost surely for any other solution $\tilde{Y}$ to (3.21).

Proof. This proof is similar to the one of Theorem 2.2.3: For any contingent claim $Y \geq \bar{x}_u$ with $Y \in L^1(P)$ and $E_P[Y] \leq x_0$ and any $c > 0$, we have

$$\begin{aligned} E_{Q_1}[-\ell(-Y)] &\leq E_{Q_1}[-\ell(-Y)] + c(x_0 - E_P[Y]) \\ &\leq E_R\left[\sup_{x>\bar{x}_u}\left\{-\frac{dQ_1}{dR}\ell(-x) - c\frac{dP}{dR}x\right\}\right] + cx_0 \\ &= -E_{Q_1}\left[\ell\left(L\left(c\frac{dP}{dQ_1}\right)\right)\right] + c\left(x_0 + E_P\left[L\left(c\frac{dP}{dQ_1}\right)\right]\right), \end{aligned} \tag{3.25}$$

where the final equality follows Lemma 3.2.12(xi).

$L\left(cdP/dQ_1\right)$ converges to $-\bar{x}_\ell$ P-almost surely as $c \to 0$ and to $-\bar{x}_u$ as $c \to \infty$. Hence by Assumption 3.2.1(d) and monotone convergence, for any $x_0 \in (\bar{x}_u, \bar{x}_\ell)$ we can find $c_{P,Q_1} > 0$ that solves (3.23). This implies $E_{Q_1}[-\ell(-Y)] \leq E_{Q_1}\left[-\ell\left(-Y_{P,Q_1}\right)\right]$ for any $Y \geq \bar{x}_u$ that satisfies the budget constraint. Y_{P,Q_1} satisfies the budget constraint and is thus a solution to (3.21).

In order to show uniqueness, let $\tilde{Y} \geq \bar{x}_u$ a loss-minimizing position that satisfies the budget constraint. Since $\ell(-x) = 0$ for $x \geq \bar{x}_\ell$, also $\tilde{Y} \cdot 1_{\{\tilde{Y} \leq \bar{x}_\ell\}}$ is a loss-minimizing position. Since ℓ is strictly convex on $[-\bar{x}_\ell, -\bar{x}_u]$ by assumption, we have $\tilde{Y} \cdot 1_{\{\tilde{Y} \leq \bar{x}_\ell\}} = Y_{P,Q_1} \cdot 1_{\{Y_{P,Q_1} \leq \bar{x}_\ell\}} = Y_{P,Q_1}$ Q_1- and hence R-almost surely. From the budget constraint $x_0 = E_P[\tilde{Y}]$ and $x_0 = E_P[Y_{P,Q_1}] = E_P[\tilde{Y} \cdot 1_{\{\tilde{Y} \leq \bar{x}_\ell\}}]$ it now follows that $\tilde{Y} \leq \bar{x}_\ell$ and hence $\tilde{Y} = Y_{P,Q_1}$ P-almost surely. Thus $\tilde{Y}$ may differ from Y_{P,Q_1} only on the set $\{dP/dR = 0\}$. □

If Assumption 3.2.1(a) holds for $\lambda_1 = 0$ and all $\lambda_2 > 0$, then there exists a unique constant $\tilde{\lambda}_2 > 0$ that solves the equation

$$x_0 \;=\; E_P\left[I\left(\tilde{\lambda}_2\frac{dP}{dQ_0}\right)\right]. \tag{3.26}$$

By Theorem 2.2.3 $I(\tilde{\lambda}_2 dP/dQ_0)$ is the unique solution to the utility maximization problem (2.22) without risk constraint.

We will now state the solution to the utility maximization problem (3.15) without model uncertainty, which is the main result of this section. Uniqueness in the following is meant in the R-almost sure sense.

Theorem 3.2.3. *Suppose that Assumption 3.2.1 holds. Let $x_1 > 0$, $x_0 > \bar{x}_u$, let c_{P,Q_1} and $\tilde{\lambda}_2$ be defined as in (3.23) and (3.26), and let Y_{P,Q_1} be the solution to the loss minimization problem (3.21) defined in (3.24). There are four cases:*

(i) We have $x_0 < \bar{x}_\ell$ and $x_1 < E_{Q_1}\left[\ell\left(-Y_{P,Q_1}\right)\right]$.

Then there is no financial position which satisfies both constraints.

(ii) We have $x_0 < \bar{x}_\ell$ and $x_1 = E_{Q_1}\left[\ell\left(-Y_{P,Q_1}\right)\right]$.

If $u\left(Y_{P,Q_1}\right)^- \in L^1(Q_0)$, then

$$\begin{aligned} X_{P,Q_1,Q_0} &:= Y_{P,Q_1} \cdot 1_{\left\{\frac{dP}{dR}>0\right\}} + \infty \cdot 1_{\left\{\frac{dP}{dR}=0\right\}} \\ &= -L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right) \cdot 1_{\left\{\frac{dP}{dR}>0\right\}} + \infty \cdot 1_{\left\{\frac{dP}{dR}=0\right\}} \end{aligned}$$

is a solution to the maximization problem (3.15), and both constraints are binding. Otherwise the maximization problem has no solution. X_{P,Q_1,Q_0} is the unique solution if $u\left(X_{P,Q_1,Q_0}\right) \in L^1(Q_0)$.

(iii) We have $E_{Q_1}[\ell(-I(\tilde{\lambda}_2 dP/dQ_0))] < x_1$. This implies that either $x_0 \geq \bar{x}_\ell$ or, if $x_0 < \bar{x}_\ell$, $x_1 > E_{Q_1}[\ell(-Y_{P,Q_1})]$.

Then

$$X_{P,Q_1,Q_0} := I\left(\tilde{\lambda}_2\frac{dP}{dQ_0}\right)$$

is the unique solution to the maximization problem (3.15), and the UBSR constraint is not binding.

(iv) We have either $x_0 \geq \bar{x}_\ell$ or, if $x_0 < \bar{x}_\ell$, $x_1 > E_{Q_1}[\ell(-Y_{P,Q_1})]$, and in both cases $E_{Q_1}[\ell(-I(\tilde{\lambda}_2 dP/dQ_0))] \geq x_1$.

Then a solution to the maximization problem (3.15) exists and both constraints are binding. The unique solution is given by

$$\begin{aligned} X_{P,Q_1,Q_0} &:= x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0}, \lambda_2^*\frac{dP}{dQ_0}\right) \\ &= \begin{cases} J\left(\lambda_1^*\frac{dQ_1}{dQ_0}, \lambda_2^*\frac{dP}{dQ_0}\right) & \text{on } \left\{\lambda_2^*\frac{dP}{dQ_0} > u'(\bar{x}_\ell) + \lambda_1^*\frac{dQ_1}{dQ_0}\ell'(-\bar{x}_\ell+)\right\} \\ \bar{x}_\ell & \text{on } \left\{u'(\bar{x}_\ell) \leq \lambda_2^*\frac{dP}{dQ_0} \leq u'(\bar{x}_\ell) + \lambda_1^*\frac{dQ_1}{dQ_0}\ell'(-\bar{x}_\ell+)\right\} \\ I\left(\lambda_2^*\frac{dP}{dQ_0}\right) & \text{on } \left\{\lambda_2^*\frac{dP}{dQ_0} < u'(\bar{x}_\ell)\right\}, \end{cases} \end{aligned}$$

where x^* *and* J *are defined as in* (3.20)*, and* $\lambda_1^* \geq 0$*,* $\lambda_2^* > 0$ *satisfy*

$$x_1 = E_{Q_1}\left[\ell\left(-X_{P,Q_1,Q_0}\right)\right] \tag{3.27}$$

and

$$x_0 = E_P\left[X_{P,Q_1,Q_0}\right]. \tag{3.28}$$

This theorem provides a complete solution to the utility maximization problem (3.15) in all possible cases. In case (i) the constraints are too strict and there exists no contingent claim that satisfies both constraints. In case (ii) the loss of the loss-minimizing contingent claim is equal to the loss threshold x_1. On the subset of Ω where P is equivalent to R the only possible investment is the one in the loss-minimizing position, and on the complement we should take X_{P,Q_1,Q_0} as large as possible. For case (iii), observe that $I(\tilde{\lambda}_2 dP/dQ_0)$ is the solution to the utility maximization problem without risk constraint. If this position satisfies the UBSR constraint, then it must also be a solution to the optimization problem with UBSR constraint.

Finally, (iv) covers all the remaining cases. In this case, the solution can be interpreted as a portfolio of an unconstrained solution under a modified budget constraint and two puts with strike $\bar{x}_\ell$, i.e.,

$$\begin{aligned} X_{P,Q_1,Q_0} = I\left(\lambda_2^* \frac{dP}{dQ_0}\right) + \left(\bar{x}_\ell - I\left(\lambda_2^* \frac{dP}{dQ_0}\right)\right)^+ \\ - \left(\bar{x}_\ell - J\left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0}\right)\right)^+ . \end{aligned} \tag{3.29}$$

The portfolio contains a long position in the asset $I\left(\lambda_2^* dP/dQ_0\right)$ which is the solution for a tighter budget constraint, but no risk constraint. Because of the UBSR constraint, an optimizing agent needs to buy insurance against portfolio values below $\bar{x}_\ell$. Here, a very conservative strategy would be the approach of a portfolio insurer who buys protection against any shortfall below the threshold $\bar{x}_\ell$. Such an agent goes long in a put on $I\left(\lambda_2^* dP/dQ_0\right)$ with strike $\bar{x}_\ell$, which guarantees full protection. In the maximization problem (3.15) the UBSR constraint is, however, not that tight. The agent can still short a put on the asset $J\left(\lambda_1^* dQ_1/dQ_0, \lambda_2^* dP/dQ_0\right)$ with strike $\bar{x}_\ell$, and gain some additional profit from selling this put. Since $J(y_1, y_2) \geq I(y_2)$ for all $(y_1, y_2) \in [0, \infty) \times (0, \infty)$, the second put in (3.29) will only be exercised if the first put is exercised. In this case, the gains from the first put are larger than the losses from the second put. Hence, going short in the second put makes our investment less costly, but we are still partly insured against losses. The final payoff is in general not bounded from below, unless the domain of the utility function is bounded from below, that is, $\bar{x}_u = 0$.

The following lemma will be used in the proof of Theorem 3.2.3 in order to ensure that the equations (3.27) and (3.28) are satisfied. Section 3.2.2 will be devoted to its proof.

Lemma 3.2.4. *Suppose that Assumption 3.2.1 holds. Let $x_1 > 0$, $x_0 > \bar{x}_u$, and let $\tilde{\lambda}_2$ be the unique solution to Equation* (3.26)*. For $x_0 < \bar{x}_\ell$, let $Y_{P,Q_1} > 0$ be the solution to the loss minimization problem defined in Proposition 3.2.2. Assume that either $x_0 \geq \bar{x}_\ell$ or, if $x_0 < \bar{x}_\ell$, $x_1 > E_{Q_1}[\ell(-Y_{P,Q_1})]$. If $E_{Q_1}[\ell(-I(\tilde{\lambda}_2 dP/dQ_0))] \geq x_1$, then there exist $\lambda_1^* \geq 0$, $\lambda_2^* > 0$ such that*

$$x_1 \;=\; E_{Q_1}\left[\ell\left(-x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0}, \lambda_2^*\frac{dP}{dQ_0}\right)\right)\right] \tag{3.30}$$

and

$$x_0 \;=\; E_P\left[x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0}, \lambda_2^*\frac{dP}{dQ_0}\right)\right]. \tag{3.31}$$

Remark 3.2.5. *If $E_{Q_1}[\ell(-I(\tilde{\lambda}_2 dP/dQ_0))] = x_1$, then $\lambda_2^* = \tilde{\lambda}_2$ and $\lambda_1^* = 0$. The solution to the constrained problem* (3.15) *equals the solution to the problem without risk constraint in this case.*

With means of this lemma we are now able to proof the main theorem of this section.

Proof of Theorem 3.2.3 . The functional $X \mapsto E_{Q_0}[u(X)]$ is strictly concave on the convex subset of $\mathcal{X}_{P,Q_1,Q_0}(x_0, x_1)$ of financial positions with finite utility. Thus, there is at most one solution to problem (3.15) Q_1- and hence R-almost surely if the utility of the optimal contingent claim is finite.

(i) This follows from Proposition 3.2.2.

(ii) X_{P,Q_1,Q_0} solves the loss minimization problem (3.21) by Proposition 3.2.2. Hence it satisfies both constraints, and by Proposition 3.2.2, any other contingent claim satisfying both constraints equals X_{P,Q_1,Q_0} on the set $\{dP/dR > 0\}$. On $\{dP/dR = 0\}$ we cannot do any better than setting X_{P,Q_1,Q_0} equal to ∞. Hence, X_{P,Q_1,Q_0} solves the utility maximization problem (3.15).

In order to show (iii) and (iv), take $X \in \mathcal{X}_{P,Q_1,Q_0}(x_0, x_1)$ and $\lambda_1 \geq 0$,

$\lambda_2 > 0$. Then

$$
\begin{aligned}
E_{Q_0}[u(X)] &\leq E_{Q_0}u[(X)] + \lambda_1(x_1 - E_{Q_1}[\ell(-X)]) + \lambda_2(x_0 - E_P[X]) \\
&\leq E_{Q_0}\left[\sup_{x>\bar{x}_u}\left\{u(x) - \lambda_1\frac{dQ_1}{dQ_0}\ell(-x) - \lambda_2\frac{dP}{dQ_0}x\right\}\right] + \lambda_1 x_1 + \lambda_2 x_0 \\
&= E_{Q_0}\left[u\left(x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right)\right] \\
&+ \quad \lambda_1\left(x_1 - E_{Q_1}\left[\ell\left(-x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right)\right]\right) \\
&+ \quad \lambda_2\left(x_0 - E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right]\right),
\end{aligned} \tag{3.32}
$$

where the equality follows from Lemma 3.2.12(x). Observe that for any $\lambda_1 \geq 0$ and $\lambda_2 > 0$, $x^*(\lambda_1 dQ_1/dQ_0, \lambda_2 dP/dQ_0) \in \mathcal{I}$ by Assumption 3.2.1(a)&(c).

(iii) First note that $E_{Q_1}[\ell(-Y_{P,Q_1})] \leq E_{Q_1}[\ell(-I(\tilde{\lambda}_2 dP/dQ_0))]$ due to Proposition 3.2.2. If $E_{Q_1}[\ell(-I(\tilde{\lambda}_2 dP/dQ_0))] < x_1$, then the last two summands in (3.32) are equal to zero for $\lambda_1 = 0$, $\lambda_2 = \tilde{\lambda}_2$. Since $x^*(0, y_2) = I(y_2)$, this implies

$$
\sup_{X \in \mathcal{X}_{P,Q_1,Q_0}(x_0,x_1)} E_{Q_0}[u(X)] \leq E_{Q_0}\left[u\left(I\left(\tilde{\lambda}_2\frac{dP}{dQ_0}\right)\right)\right].
$$

Thus, $I(\tilde{\lambda}_2 dP/dQ_0)$ is a solution, and the UBSR constraint is not binding. Uniqueness follows from Assumption 3.2.1(c) for $\lambda_1 = 0$.

(iv) By Lemma 3.2.4 there exist $\lambda_1^* \geq 0$ and $\lambda_2^* > 0$ such that the last two summands in (3.32) are equal to zero. This implies that X_{P,Q_1,Q_0} satisfies the constraint and that

$$
\sup_{X \in \mathcal{X}_{P,Q_1,Q_0}(x_0,x_1)} E_{Q_0}[u(X)] \leq E_{Q_0}\left[u\left(X_{P,Q_1,Q_0}\right)\right].
$$

Hence, X_{P,Q_1,Q_0} is a solution to problem (3.15), and both constraints are binding. Uniqueness follows from Assumption 3.2.1(c). □

3.2.1 Duality Results

In this section we complement the solutions from the previous section by some duality results. They will be useful when we solve the general robust problem in an incomplete market.

Let us extend the usual definition of the convex conjugate of a utility function from Chapter 2 in order to incorporate the second constraint. Define the convex function

$$v(y_2, y_1, y_0) := \sup_{x > \bar{x}_u} \{y_0 u(x) - y_1 \ell(-x) - y_2 x\}$$

on $(0, \infty) \times [0, \infty) \times (0, \infty)$. For the problem of loss minimization, we define a convex conjugate $\tilde{v}$ of the loss function. Here we start directly with the function $\tilde{v}(\cdot, \cdot)$ on $\mathbb{R}^2$, i.e., we define

$$\tilde{v}(y_2, y_1) := \sup_{x > \bar{x}_u} \{-y_1 \ell(-x) - y_2 x\}$$

on $(0, \infty) \times [0, \infty)$; compare Remark 2.1.9.

Let $Q_0 \in \mathcal{Q}_0$, $Q_1 \in \mathcal{Q}_1$, and $P \in \mathcal{P}'$, where $\mathcal{P}' = \mathcal{P}$ if $\bar{x}_u = -\infty$ and $\mathcal{P}' = \mathcal{P}^T$ if $\bar{x}_u = 0$. For $\lambda_1 \geq 0$ and $\lambda_2 > 0$, let us define the convex functionals

$$v_{\lambda_1, \lambda_2}(P|Q_1|Q_0) := E_R \left[v \left(\lambda_2 \frac{dP}{dR}, \lambda_1 \frac{dQ_1}{dR}, \frac{dQ_0}{dR} \right) \right] \tag{3.33}$$

and

$$\tilde{v}_c(P|Q_1) := E_R \left[\tilde{v} \left(c \frac{dP}{dR}, \frac{dQ_1}{dR} \right) \right]. \tag{3.34}$$

By Lemma 3.2.12(x)&(xi) we have

$$\begin{aligned} v_{\lambda_1, \lambda_2}(P|Q_1|Q_0) = E_{Q_0} &\left[u \left(x^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right) \right] \\ &- \lambda_1 E_{Q_1} \left[\ell \left(-x^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right) \right] \\ &- \lambda_2 E_P \left[x^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right] \end{aligned}$$

and

$$\tilde{v}_c(P|Q_1) = -E_{Q_1} \left[\ell \left(L \left(c \frac{dP}{dQ_1} \right) \right) \right] + c E_P \left[L \left(c \frac{dP}{dQ_1} \right) \right],$$

and these two properties will be crucial for our duality results and the solution of the general robust utility maximization problem under a joint budget and loss constraint.

Remark 3.2.6. *(i) $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0)$ and $\tilde{v}_c(P|Q_1)$ are well defined (possibly infinite). Indeed, note that v and $\tilde{v}$ are decreasing in y_2 if $\bar{x}_u = 0$ and recall that $P(\Omega) = 1$ if $\bar{x}_u = -\infty$. Hence, due to the definitions of v and $\tilde{v}$ as suprema of linear functions we have for any $\lambda_1 \geq 0$ and $\lambda_2, c > 0$*

$$\begin{aligned} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) &\geq \sup_{x>\bar{x}_u} E_R\left[\frac{dQ_0}{dR}u(x) - \lambda_1\frac{dQ_1}{dR}\ell(-x) - \lambda_2\frac{dP}{dR}x\right] \\ &= v(\lambda_2 P(\Omega), \lambda_1, 1) \geq v(\lambda_2, \lambda_1, 1) > -\infty \end{aligned}$$

and

$$\tilde{v}_c(P|Q_1) \geq \sup_{x>\bar{x}_u} E_R\left[-\frac{dQ_1}{dR}l(-x) - c\frac{dP}{dR}x\right] = \tilde{v}(cP(\Omega), 1) \geq \tilde{v}(c, 1) > -\infty.$$

(ii) If $\bar{x}_u = 0$, then $\tilde{v}(y_2, y_1) \leq 0$ for all $y_1 \geq 0$ and $y_2 > 0$ and hence

$$\tilde{v}_c(P|Q_1) \leq 0.$$

Analogously to Assumption 2.29 we now need

Assumption 3.2.7. *We suppose that*

(i)

$$v_{0,1}(P|Q_1|Q_0) < \infty \tag{3.35}$$

and that

(ii)

$$\tilde{v}_1(P|Q_1) < \infty. \tag{3.36}$$

Note that (3.35) is in fact the same requirement on the measures P and Q_0 as in Chapter 2, since we have $v_{0,1}(P|Q_1|Q_0) = v(P|Q_0)$, where the right-hand side is the v-divergence of P with respect to Q_0 defined in (2.19).

Remark 3.2.8. *(i) Due to the Assumption* (2.9) *of reasonable asymptotic elasticity on the utility function u, Assumption 3.2.7(i) implies*

$$v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) < \infty \quad \textit{for all } \lambda_1 \geq 0\ \lambda_1 > 0. \tag{3.37}$$

Indeed, by Lemma 2.1.6(iv) there are functions $a > 0$ and $b \geq 0$ such that for $\lambda_2 > 0$ and $y_2, y_0 > 0$

$$v(\lambda_2 y_2, 0, y_0) \leq a(\lambda_2)v(y_2, 0, y_0) + b(\lambda_2)(y_2 + 1).$$

Since v is decreasing in y_1, (3.37) *now follows from* (3.35).

(ii) If $\bar{x}_u = 0$*, then* (3.37) *is even equivalent to*

$$v_{1,1}(P|Q_1|Q_0) < \infty.$$

Indeed, in this case we have

$$\begin{aligned} v(\lambda_2 y_2, \lambda_1 y_1, y_0) &\le v(\lambda_2 y_2, 0, y_0) \\ &\le a(\lambda_2) v(y_2, 0, y_0) + b(\lambda_2)(y_2 + 1) \\ &\le a(\lambda_2)(v(y_2, y_1, y_0) + y_1 \ell(0)) + b(\lambda_2)(y_2 + 1) \end{aligned}$$

for $\lambda_1 \ge 0$ *and* $\lambda_2 > 0$*.*

(iii) If $\bar{x}_u = -\infty$*, then due to the assumption of reasonable asymptotic elasticity* (3.13) *on the loss function* ℓ *we obtain as above the equivalence of* (3.36) *and*

$$\tilde{v}_c(P|Q_1) < \infty \quad \text{for all } c > 0.$$

(iv) If $\bar{x}_u = 0$*, then* $\tilde{v}(y_2, y_1) \le 0$*, and* (3.36) *is always satisfied.*

The following lemma is the analogue result to Lemma 2.2.1. It shows that Assumption 3.2.7 is equivalent to the integrability assumptions that were needed for the solution of the primal utility maximization problem (3.15) without model uncertainty. Furthermore, it will be necessary for the solution of the general robust utility maximization problem in Section 3.3 below.

Lemma 3.2.9. *(i) Assumptions 3.2.1(a)-(c) and 3.2.7(i) are equivalent.*

(ii) If $\bar{x}_u = 0$*, then Assumption 3.2.1(d)&(e) is always satisfied.*

(iii) If $\bar{x}_u = -\infty$*, then Assumption 3.2.1(d)&(e) is equivalent to 3.2.7(ii).*

Proof. This proof is similar to the one of Lemma 2.2.1.

(i) By Lemma 3.2.12(x) v is continuously differentiable in $y_1 \ge 0$ and $y_2 > 0$. We will first show that Assumption 3.2.7(i) implies (a)-(c) of Assumption 3.2.1.

(a) Let first $\lambda_1 \ge 0$ be fixed and define $y_0 := dQ_0/dR > 0$, $y_1 := \lambda_1 dQ_1/dR \ge 0$, $\phi := dP/dR$, $y_2 := \lambda_2 \phi$ for $\lambda_2 > 0$, and the function

$$f(y_2) := v(y_2, y_1, y_0).$$

Since f is convex, we obtain for $0 < \mu < \nu$,

$$f(\nu\phi) - f((\nu - \mu)\phi) \le \mu\phi f'(\nu\phi) \le f((\nu + \mu)\phi) - f(\nu\phi)$$

on $\{\phi > 0\}$. If $f(0) = u(\infty) - \lambda_1 \ell(-\infty) < \infty$, then the three parts of the above inequality are equal to zero for $\phi = 0$. Otherwise $\phi > 0$ R-almost surely since $E_R[f(\phi)] < \infty$. Hence

$$E_R\left[f\left(\nu\phi\right)\right] - E_R\left[f\left((\nu-\mu)\phi\right)\right] \le \mu E_P\left[f'\left(\nu\phi\right)\right]$$
$$\le E_R\left[f\left((\nu+\mu)\phi\right)\right] - E_R\left[f\left(\nu\phi\right)\right].$$

By Lemma 3.2.12(x), $f'(y_2) = -x^*\left(y_1/y_0, y_2/y_0\right)$. Multiplying all parts by -1 thus leads to

$$v_{\lambda_1,\nu}(P|Q_1|Q_0) - v_{\lambda_1,\nu+\mu}(P|Q_1|Q_0) \le \mu E_P\left[x^*\left(\lambda_1 \frac{dQ_1}{dQ_0}, \nu\frac{dP}{dQ_0}\right)\right]$$
$$\le v_{\lambda_1,\nu-\mu}(P|Q_1|Q_0) - v_{\lambda_1,\nu}(P|Q_1|Q_0).$$

By Remark 3.2.6(i) $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) > -\infty$ for all $\lambda_1 \ge 0, \lambda_2 > 0$. By Remark 3.2.8(i) $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) < \infty$ for all $\lambda_1 \ge 0, \lambda_2 > 0$ if Assumption 3.2.7 holds. Hence the right-hand and the left-hand side in the above inequality are finite, which implies 3.2.1(a).

3.2.7(b) follows with $\partial v(y_2, y_1, y_0)/\partial y_1 = -\ell\left(-x^*\left(y_1/y_0, y_2/y_0\right)\right)$ analogously.

(c) Finally, we obtain from Lemma 3.2.12(x)

$$\begin{aligned} \frac{dQ_0}{dR} u\left(x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right) \;&=\; v\left(\lambda_2\frac{dP}{dR}, \lambda_1\frac{dQ_1}{dR}, \frac{dQ_0}{dR}\right) \\ &+ \lambda_1\frac{dQ_1}{dR}\ell\left(-x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right) \;+\; \lambda_2\frac{dP}{dR}x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right). \end{aligned} \tag{3.38}$$

Since we just showed that the right-hand side is in $L^1(R)$, 3.2.7(c) is also proven.

On the other hand, (3.38) shows that Assumption 3.2.1(a)-(c) implies Assumption 3.2.7(i).

(ii) Due to $0 \le \ell(x) \le \ell(-\bar{x}_u) = \ell(0)$ for all $x \ge \bar{x}_u$, Assumption 3.2.1(e) is satisfied. Furthermore, it follows from Lemma 3.2.12(xi) that

$$\tilde{v}\left(c\frac{dP}{dR}, \frac{dQ_1}{dR}\right) = -\frac{dQ_1}{dR}\ell\left(L\left(c\frac{dP}{dQ_1}\right)\right) + c\frac{dP}{dR}L\left(c\frac{dP}{dQ_1}\right).$$

The expectation of the left-hand side is non-positive and finite due to Remark 3.2.6(i), $0 \le \ell \circ L \le \ell(0)$, hence Assumption 3.2.1(d) is always satisfied.

(iii) follows from Lemma 3.2.12(xi) in the same way as (i). □

The following Lemma parallels the results of Lemma 2.2.2. It will be used for the characterization of the solutions λ_1^* and λ_2^* to Equations (3.30) and (3.31) as minimizers of certain convex functions.

Lemma 3.2.10. *Let Assumption 3.2.7 hold, i.e., assume that* $v_{0,1}(P|Q_1|Q_0)$ *and* $\tilde{v}_1(P|Q_1)$ *are finite. Then* $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0)$ *is continuously differentiable in* $\lambda_1 \geq 0$ *and* $\lambda_2 > 0$ *with*

$$\frac{\partial}{\partial \lambda_1} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) = -E_{Q_1}\left[\ell\left(-x^*\left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0}\right)\right)\right] \tag{3.39}$$

and

$$\frac{\partial}{\partial \lambda_2} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) = -E_P\left[x^*\left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0}\right)\right]. \tag{3.40}$$

Furthermore, $\tilde{v}_c(P|Q_1)$ *is continuously differentiable in* $c > 0$ *with*

$$\frac{\partial}{\partial c} \tilde{v}_c(P|Q_1) = E_P\left[L\left(c \frac{dP}{dQ_1}\right)\right]. \tag{3.41}$$

Proof. By Lemma 3.2.12(x)&(xi) v and $\tilde{v}$ are continuously differentiable with

$$\frac{\partial}{\partial y_1} v(y_2, y_1, y_0) = -\ell\left(-x^*\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right)\right),$$

$$\frac{\partial}{\partial y_2} v(y_2, y_1, y_0) = -x^*\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right),$$

and

$$\frac{\partial}{\partial y_2} \tilde{v}(y_2, y_1) = L\left(\frac{y_2}{y_1}\right).$$

By Lemma 3.2.9

$$x^*\left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0}\right) \in L^1(P),$$

$$\ell\left(-x^*\left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0}\right)\right) \in L^1(Q_1),$$

and

$$L\left(c \frac{dP}{dQ_1}\right) \in L^1(P)$$

for any $\lambda_1 \geq 0$, $\lambda_2 > 0$, and $c > 0$. Furthermore, x^* is decreasing in y_2, $\ell \circ (-x^*)$ is decreasing in y_1, and L is increasing. Hence we may use Fubini's theorem to obtain for $0 < \lambda_2^1 < \lambda_2^2$

$$\begin{aligned} v_{\lambda_1,\lambda_2^2}(P|Q_1|Q_0) &= v_{\lambda_1,\lambda_2^1}(P|Q_1|Q_0) - E_R\left[\int_{\lambda_2^1}^{\lambda_2^2} x^*\left(\lambda_1\frac{dQ_1}{dQ_0},\nu\frac{dP}{dQ_0}\right)\frac{dP}{dR}d\nu\right] \\ &= v_{\lambda_1,\lambda_2^1}(P|Q_1|Q_0) - \int_{\lambda_2^1}^{\lambda_2^2} E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0},\nu\frac{dP}{dQ_0}\right)\right]d\nu, \end{aligned}$$

and for $0 \leq \lambda_1^1 < \lambda_1^2$

$$\begin{aligned} &v_{\lambda_1^2,\lambda_2}(P|Q_1|Q_0) \\ &\quad= v_{\lambda_1^1,\lambda_2}(P|Q_1|Q_0) - E_R\left[\int_{\lambda_1^1}^{\lambda_1^2} \ell\left(-x^*\left(\nu\frac{dQ_1}{dQ_0},\lambda_2\frac{dP}{dQ_0}\right)\right)\frac{dQ_1}{dR}d\nu\right] \\ &\quad= v_{\lambda_1^1,\lambda_2}(P|Q_1|Q_0) - \int_{\lambda_1^1}^{\lambda_1^2} E_{Q_1}\left[\ell\left(-x^*\left(\nu\frac{dQ_1}{dQ_0},\lambda_2\frac{dP}{dQ_0}\right)\right)\right]d\nu, \end{aligned}$$

and for $0 < c^1 < c^2$

$$\begin{aligned} \tilde{v}_{c^2}(P|Q_1) &= \tilde{v}_{c^1}(P|Q_1) + E_R\left[\int_{c^1}^{c^2} L\left(\nu\frac{dP}{dQ_1}\right)\frac{dP}{dR}d\nu\right] \\ &= \tilde{v}_{c^1}(P|Q_1) + \int_{c^1}^{c^2} E_P\left[L\left(\nu\frac{dP}{dQ_1}\right)\right]d\nu. \end{aligned}$$

This completes the proof. $\square$

Lemma 3.2.10 implies the following corollary, which will be useful to show the existence of a pair of Lagrange multiplies that satisfy the constraints also in the robust case. Furthermore, it links the utility maximization and the loss minimization problems to their corresponding dual problems, and it provides an alternative way to Lemma 2.3.4 of determining the Lagrange multipliers λ_1^* and λ_2^*. Recall that λ_1^* and λ_2^* were defined as the solution to the equations (3.30) and (3.31), i.e.,

$$x_1 = E_{Q_1}\left[\ell\left(-x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0},\lambda_2^*\frac{dP}{dQ_0}\right)\right)\right] \tag{3.42}$$

and

$$x_0 = E_P\left[x^*\left(\lambda_1^*\frac{dQ_1}{dQ_0},\lambda_2^*\frac{dP}{dQ_0}\right)\right]. \tag{3.43}$$

Furthermore, c_{P,Q_1} is the solution to Equation (3.23), i.e.,

$$x_0 = -E_P \left[L \left(c_{P,Q_1} \frac{dP}{dQ_1} \right) \right]. \tag{3.44}$$

The corresponding result without risk constraint can be found in Lemma 2.3.1.

Corollary 3.2.11. *Suppose that Assumption 3.2.7 holds. Let X_{P,Q_1,Q_0} and Y_{P,Q_1} be the solutions to the utility maximization problem* (3.15) *and the loss minimization problem* (3.21) *defined in Theorem 3.2.3 and Proposition 3.2.2, respectively.*

(i) Assume that either $x_0 \geq \bar{x}_\ell$ or, if $x_0 < \bar{x}_\ell$, $x_1 > E_{Q_1}[\ell(-Y_{P,Q_1})]$. Conditions (3.42) *and* (3.43) *are satisfied if and only if $(\lambda_1^*, \lambda_2^*)$ minimize*

$$v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0$$

over $\lambda_1 \geq 0$ and $\lambda_2 > 0$. Furthermore, we have

$$E_{Q_0}[u(X_{P,Q_1,Q_0})] = v_{\lambda_1^*,\lambda_2^*}(P|Q_1|Q_0) + \lambda_1^* x_1 + \lambda_2^* x_0. \tag{3.45}$$

(ii) Let $x_0 \in (\bar{x}_u, \bar{x}_\ell)$. Condition (3.44) *is satisfied if and only if c_{P,Q_1} minimizes*

$$\tilde{v}_c(P|Q_1) + c x_0$$

over $c > 0$. In this case we have

$$-E_{Q_1}[\ell(-Y_{P,Q_1})] = \tilde{v}_{c_{P,Q_1}}(P|Q_1) + c_{P,Q_1} x_0.$$

Proof. This follows immediately from Lemma 3.2.10 and from the proofs of Proposition 3.3.6 and Theorem 3.2.3:

(i) Since the convex function $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0$ is continuously differentiable, it assumes its infimum in $(\lambda_1^*, \lambda_2^*)$ if and only if we have $\partial v_{\lambda_1,\lambda_2}(P|Q_1|Q_0)/\partial\lambda_1 + x_1 = 0$ and $\partial v_{\lambda_1,\lambda_2}(P|Q_1|Q_0)/\partial\lambda_2 + x_2 = 0$ for $\lambda_1 = \lambda_1^*$ and $\lambda_2 = \lambda_2^*$. Due to Lemma 3.2.10 this is equivalent to (3.42) and (3.43). In order to show (3.45), note that Inequality (3.32) holds as equality if λ_1 and λ_2 satisfy (3.42) and (3.43) and $X = X_{P,Q_1,Q_0}$.

(ii) This is shown in the same way using (3.41) and Inequality (3.25). □

3.2.2 Auxiliary Results

In this section we collect properties of the deterministic functions x^* and L and prove Lemma 3.2.4.

Properties of the Deterministic Functions x^* and L

Here we will discuss how the functions x^* and L, that give us the optimal and the loss minimizing contingent claims, can be obtained and describe their properties. For this purpose we consider a family of functions g_{y_1,y_2} with $y_1, y_2 \geq 0$, defined by

$$g_{y_1,y_2}(x) := u(x) - y_1\ell(-x) - y_2 x.$$

In the following we will sometimes drop the indices y_1, y_2 if there is no danger of confusion.

Lemma 3.2.12.

(i) g_{y_1,y_2} *is strictly concave and thus continuous on its essential domain*

$$dom(g_{y_1,y_2}) = dom(u).$$

(ii) g_{y_1,y_2} *attains its supremum on* $\mathbb{R}$ *if and only if* $y_2 > 0$. *In this case, the maximizer is unique and equals*

$$x^*(y_1,y_2) \quad := \quad \begin{cases} J(y_1,y_2) & \text{if } y_2 > u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+), \\ \bar{x}_\ell & \text{if } u'(\bar{x}_\ell) \leq y_2 \leq u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+), \\ I(y_2) & \text{if } y_2 < u'(\bar{x}_\ell). \end{cases} \tag{3.46}$$

Here $J(y_1,y_2)$ *denotes the unique solution to* $u'(x) + y_1\ell'(-x) = y_2$ *for the case that* $y_2 > u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)$, *and* $I := (u')^{-1}$.

(iii) If $\bar{x}_\ell = \infty$, (3.46) *simplifies to*

$$x^*(y_1,y_2) = J(y_1,y_2).$$

(iv) The function $x^* : [0,\infty) \times (0,\infty) \to (\bar{x}_u, \infty)$, *defined in* (3.46), *is continuous.*

(v) $x^*(y_1,y_2)$ *is decreasing in* y_2 *for* $y_1 \geq 0$ *fixed, and increasing in* y_1 *for* $y_2 > 0$ *fixed.*

(vi) For fixed $y_1 \geq 0$, *we have* $x^*(y_1,\infty) := \lim_{y_2\to\infty} x^*(y_1,y_2) = \bar{x}_u \in \mathbb{R}\cup\{-\infty\}$ $x^*(y_1,0) := \lim_{y_2\to 0} x^*(y_1,y_2) = \infty$.

(vii) If $\alpha \geq 1$, *then* $x^*(\alpha y_1, \alpha y_2) \leq x^*(y_1,y_2)$.

(viii) *Let* $L : \mathbb{R} \to [-\bar{x}_\ell, -\bar{x}_u]$ *be the generalized inverse of the derivative of the loss function* ℓ, *i.e.,*

$$L(y) := \begin{cases} -\bar{x}_u & \text{if } y \geq \ell'(-\bar{x}_u), \\ (\ell')^{-1}(y) & \text{if } \ell'(-\bar{x}_\ell+) < y < \ell'(-\bar{x}_u), \\ -\bar{x}_\ell & \text{if } y \leq \ell'(-\bar{x}_\ell+). \end{cases} \tag{3.47}$$

L *is a continuous function which is strictly increasing on the interval* $[\ell'(-\bar{x}_\ell+), \ell'(-\bar{x}_u)]$.

If $e > 0$ *is such that* $\ell'(-\bar{x}_\ell+) < e < \ell'(-\bar{x}_u)$, *and* $\mu := u'(-L(e))$, *then we have for all* $y_1 \geq 0$,

$$x^*(0, \mu) = x^*(y_1, \mu + y_1 e).$$

(ix) *Let* $\tilde{c} : \mathbb{R}_+ \to \mathbb{R}_+$ *be decreasing with* $\lim_{y_1 \to \infty} \tilde{c}(y_1) = c > 0$. *Then*

$$\lim_{y_1 \to \infty} x^*(y_1, \tilde{c}(y_1) \cdot y_1) \;=\; -L(c) \;\in\; [\bar{x}_u, \bar{x}_\ell].$$

Moreover, $x^*(y_1, cy_1)$ *converges for* $y_1 \to \infty$ *to* $-L(c)$ *monotonously from above.*

(x) *Define*

$$\begin{aligned} v(y_2, y_1, y_0) :&= \sup_{x > \bar{x}_u} \{y_0 u(x) - y_1 \ell(-x) - y_2 x\} \\ &= y_0 u\left(x^*\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right)\right) - y_1 \ell\left(-x^*\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right)\right) \\ &\quad - y_2 x^*\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right) \end{aligned} \tag{3.48}$$

for $y_2 > 0$, $y_1 \geq 0$, *and* $y_0 > 0$. v *is convex and continuously differentiable with derivatives*

$$\frac{\partial}{\partial y_0} v(y_2, y_1, y_0) = u\left(x^*\left(\frac{y_1}{y_0}, \frac{y_1}{y_0}\right)\right), \tag{3.49}$$

$$\frac{\partial}{\partial y_1} v(y_2, y_1, y_0) = -\ell\left(-x^*\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right)\right), \tag{3.50}$$

and

$$\frac{\partial}{\partial y_2} v(y_2, y_1, y_0) = -x^*\left(\frac{y_1}{y_0}, \frac{y_2}{y_0}\right). \tag{3.51}$$

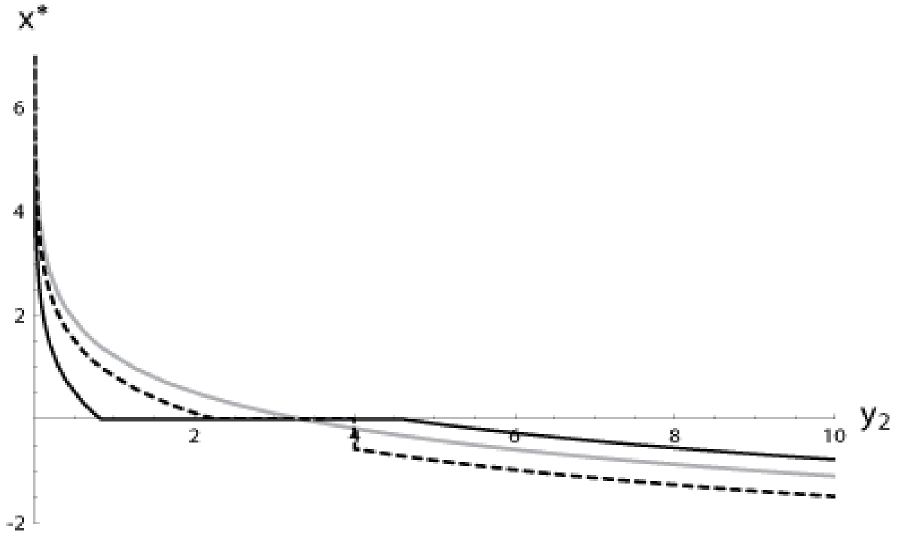

Figure 3.1: x^* as a function of y_2

Hence v is decreasing in y_1, and it is decreasing in y_2 if $\bar{x}_u = 0$. Furthermore,

$$\begin{aligned} v(0, y_1, y_0) &:= \lim_{y_2 \to 0} v(y_2, y_1, y_0) \\ &= y_0 u(\infty) - y_1 \ell(-\infty) := \lim_{x\to\infty} (y_0 u(x) - y_1 \ell(-x)) \end{aligned}$$

for $y_1 \geq 0$, $y_0 > 0$.

(xi) Define

$$\tilde{v}(y_2, y_1) := \sup_{x > \bar{x}_u} \{-y_1 \ell(-x) - y_2 x\} = -y_1 \ell\left(L\left(\frac{y_2}{y_1}\right)\right) + y_2 L\left(\frac{y_2}{y_1}\right) \tag{3.52}$$

for $y_2 > 0$ and $y_1 > 0$. $\tilde{v}$ is convex and continuously differentiable with derivatives

$$\frac{\partial}{\partial y_1}\tilde{v}(y_2, y_1) = -\ell\left(L\left(\frac{y_2}{y_1}\right)\right),$$

and

$$\frac{\partial}{\partial y_2}\tilde{v}(y_2, y_1) = L\left(\frac{y_2}{y_1}\right).$$

Figure 3.1 shows an example of $x^*(\lambda_1^* y_1, \lambda_2^* y_2)$ as a function of y_2, where λ_1^* and λ_2^* are the parameters from Theorem 3.2.3 such that X_{P,Q_1,Q_0} satisfies the constraints. We choose the exponential utility function $u(x) = 1 - e^{-x}$. The black line shows x^*, where $\ell(x) = (e^x - e^{-\bar{x}_\ell}) \vee 0$ with $\bar{x}_\ell = 0$. The gray line shows $x^*(0, \tilde{\lambda}_2 y_2) = I(\tilde{\lambda}_2)$, which gives the optimal contingent claim without risk constraint. The dashed line shows the deterministic solution with a VaR constraint. For the latter case, the solution can be found in Basak and Shapiro [2001].

Proof of Lemma 3.2.12. (i) The sum of the strictly concave function u and the concave function $-\ell(-\cdot)$ is strictly concave, and the domain of g is $\mathrm{dom}(g) = \mathrm{dom}(u) \cap \mathrm{dom}(\ell(-\cdot)) = \mathrm{dom}(u)$.

(ii)&(iii) Suppose first that $y_2 = 0$. Then $g(x) = u(x) - y_1 \ell(-x)$, and g does not attain its supremum on $\mathbb{R}$. If conversely $y_2 > 0$, then

$$g'(x) = \begin{cases} u'(x) + y_1 \ell'(-x) - y_2 & \text{if } x < \bar{x}_\ell, \\ u'(x) - y_2 & \text{if } x > \bar{x}_\ell. \end{cases}$$

Hence by the Inada conditions (2.7) and (2.8) we have

$$\lim_{x \searrow \bar{x}_u} g'(x) = \infty > 0 \qquad \text{and} \qquad \lim_{x \to \infty} g'(x) = -y_2 < 0$$

because $\ell' \geq 0$ and $\lim_{x \to \infty} \ell'(-x) = 0$ if $\bar{x}_\ell = \infty$. Since g is strictly concave on its essential domain, this implies that g has a unique maximum.

Next we prove that the maximizer of g is given by x^* as defined in (3.46). Suppose first that $\bar{x}_\ell < \infty$.

If $y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +)$, then $g'(\bar{x}_\ell -) < 0$. It follows that g is decreasing in a neighborhood of $\bar{x}_\ell$. Thus, $x^*(y_1, y_2) < \bar{x}_\ell$. Since g is strictly concave and continuously differentiable on the interval $(\bar{x}_u, \bar{x}_\ell)$, $x^*(y_1, y_2)$ is characterized as the unique solution of $g'(x) = 0$ with $x \in (\bar{x}_u, \bar{x}_\ell)$. This implies that $x^*(y_1, y_2) = J(y_1, y_2)$.

If $y_2 < u'(\bar{x}_\ell)$, then $g'(\bar{x}_\ell +) > 0$. It follows that g is increasing in a neighborhood of $\bar{x}_\ell$. Thus, $x^*(y_1, y_2) > \bar{x}_\ell$. In this case, the first order condition implies that $x^*(y_1, y_2) = I(y_2)$.

If $u'(\bar{x}_\ell) \leq y_2 \leq u'(\bar{x}_\ell) + y_2 \ell'(-\bar{x}_\ell +)$, then $g'(\bar{x}_\ell -) \geq 0 \geq g'(\bar{x}_\ell +)$. Since g is strictly concave, we obtain that $x^*(y_1, y_2) = \bar{x}_\ell$.

Next, let us assume that $\bar{x}_\ell = \infty$. Then $g(x) = u(x) - y_1 \ell(-x) - y_2 x$ for all $x \in \mathrm{dom}(g)$, thus $x^*(y_1, y_2) = J(y_1, y_2)$ by the first order condition which proves (iii). Moreover, by our assumptions on u and ℓ, the condition $y_2 > u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +)$ is trivially satisfied in this case.

(iv) Since the inverse of a continuous and strictly decreasing function is continuous, J is continuous on $\{(y_1, y_2) : \; y_2 \geq u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +), y_1 \geq 0, y_2 > 0\}$, and I is continuous on $(0, u'(\bar{x}_\ell)]$. Furthermore, $J(y_1, y_2) = \bar{x}_\ell$ if

$y_2 = u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)$, and $I(y_2) = \bar{x}_\ell$ if $y_2 = u'(\bar{x}_\ell)$. Altogether, it follows that x^* is a continuous function.

(v) Simply note that both $u'(x)$ and $\ell'(-x)$ are decreasing in x.

(vi) For $y_1 \geq 0$ fixed, $u'(x)+y_1\ell'(-x)$ is strictly decreasing and continuous in x on the interval $(\bar{x}_u, \bar{x}_\ell)$ with $\lim_{x\searrow\bar{x}_u} (u'(x) + y_1\ell'(-x)) = \infty$. This implies $x^*(y_1, y_2) \to \bar{x}_u$ as $y_2 \to \infty$.

Moreover, $\lim_{y_2\to 0} x^*(y_1, y_2) = \lim_{y_2\to 0} I(y_2) = \infty$.

(vii) We first show the claim for $y_2 \geq u'(\bar{x}_\ell)+y_1\ell'(-\bar{x}_\ell+)$. Since $x^*(y_1, y_2) = \bar{x}_\ell$ for $y_2 = u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)$ and $x^*(y_1, y_2) \leq \bar{x}_\ell$ for $y_2 > u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)$, we may restrict our attention to $y_2 > u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)$. Then

$$\alpha y_2 > \alpha u'(\bar{x}_\ell) + \alpha y_1\ell'(-\bar{x}_\ell+) \geq u'(\bar{x}_\ell) + \alpha y_1\ell'(-\bar{x}_\ell+).$$

Thus, $x^*(y_1, y_2)$ is the unique solution of the equation $u'(x) + y_1\ell'(-x) = y_2$, and $x^*(\alpha y_1, \alpha y_2)$ is the unique solution of $u'(x) + \alpha y_1\ell'(-x) = \alpha y_2$. This implies

$$\begin{aligned}\alpha y_2 &= \alpha u'(x^*(y_1, y_2)) + \alpha y_1\ell'(-x^*(y_1, y_2))\\ &> u'(x^*(y_1, y_2)) + \alpha y_1\ell'(-x^*(y_1, y_2)).\end{aligned}$$

Since $u'(x)$ and $\ell'(-x)$ are decreasing in x on $(\bar{x}_u, \bar{x}_\ell)$, we obtain $x^*(\alpha y_1, \alpha y_2) \leq x^*(y_1, y_2)$.

If $y_2 \leq u'(\bar{x}_\ell)+y_1\ell'(-\bar{x}_\ell+)$, $x^*(y_1, y_2)$ depends on y_2 only and is decreasing in y_2. Now the result follows easily.

(viii) The properties of L follow immediately from our assumptions on ℓ.

In order to derive the last claim, observe that $x^*(0, \mu)$ is the unique solution of $u'(x) = \mu$ or, equivalently, $x = I(\mu)$. If $\ell'(-\bar{x}_\ell+) < e < \ell'(-\bar{x}_u)$, then $\mu = u'(-L(e)) > u'(\bar{x}_\ell)$. Thus, $\mu + y_1 e > u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)$. This implies that $x^*(y_1, \mu + y_1 e)$ is the unique solution to $u'(x) + y_1\ell'(-x) = \mu + y_1 e$. On the other hand, since $e < \ell'(-\bar{x}_u)$,

$$\begin{aligned}u'(x^*(0, \mu)) + y_1\ell'(-x^*(0, \mu)) &= \mu + y_1\ell'(-I(\mu))\\ &= \mu + y_1\ell'[-I(u'(-L(e)))]\\ &= \mu + y_1 e,\end{aligned}$$

and $x^*(0, \mu)$ is also the unique solution to $u'(x) + y_1\ell'(-x) = \mu + y_1 e$. Thus, $x^*(0, \mu) = x^*(y_1, \mu + y_1 e)$.

(ix) If $c \geq \ell'(-\bar{x}_u)$, then $\tilde{c}(y_1) \geq c > u'(\bar{x}_\ell)/y_1 + \ell'(-\bar{x}_\ell+)$ for y_1 large enough because $\ell'(-\bar{x}_u) > \ell'(-\bar{x}_\ell+)$. Therefore, $x^*(y_1, \tilde{c}(y_1)y_1)$ satisfies

$$u'(x^*(y_1, \tilde{c}(y_1)y_1)) + y_1\ell'(-x^*(y_1, \tilde{c}(y_1)y_1)) = \tilde{c}(y_1)y_1$$

for y_1 large enough. Due to $\tilde{c}(y_1) \geq c \geq \ell'(-\bar{x}_u)$, this implies

$$u'(x^*(y_1, \tilde{c}(y_1)y_1)) \geq y_1[\ell'(-\bar{x}_u) - \ell'(-x^*(y_1, \tilde{c}(y_1)y_1))]$$

and hence $\lim_{y_1\to\infty} x^*(y_1, \tilde{c}(y_1)y_1) = \bar{x}_u = -L(c)$ due to the Inada condition (2.8) and since ℓ' is strictly increasing in $-\bar{x}_u$.

Now assume that $c < \ell'(-\bar{x}_u)$. We show that $x^*(y_1, \tilde{c}(y_1)y_1)$ is bounded from below away from $\bar{x}_u$ for large enough y_1. For this purpose, choose e such that $\ell'(-\bar{x}_\ell+) < e < \ell'(-\bar{x}_u)$ and $e > \tilde{c}(y_1)$ for y_1 large enough. It follows from (viii) that for all such y_1 we have

$$\bar{x}_u < x^*(0, \mu) = x^*(y_1, \mu + y_1 e) \leq x^*(y_1, \mu + \tilde{c}(y_1)y_1) \leq x^*(y_1, \tilde{c}(y_1)y_1),$$

where $\mu := u'(-L(e))$. This proves boundedness from below.

For y_1 large enough, we have $\tilde{c}(y_1)y_1 \geq u'(\bar{x}_\ell)$. For any such y_1, we distinguish two cases. If $\tilde{c}(y_1)y_1 \leq u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)$, then $x^*(y_1, \tilde{c}(y_1)y_1) = \bar{x}_\ell \overset{(*)}{=} -L(z(y_1))$, where

$$z(y_1) \ := \ \tilde{c}(y_1) - \frac{u'(x^*(y_1, \tilde{c}(y_1)y_1))}{y_1}.$$

Equation $(*)$ can easily be checked, since $x^*(y_1, \tilde{c}(y_1)y_1) = \bar{x}_\ell$.

If $\tilde{c}(y_1)y_1 > u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)$, then

$$u'(x^*(y_1, \tilde{c}(y_1)y_1)) + y_1\ell'(-x^*(y_1, \tilde{c}(y_1)y_1)) = \tilde{c}(y_1)y_1,$$

thus $x^*(y_1, \tilde{c}(y_1)y_1) = -L(z(y_1))$.

Since $x^*(y_1, \tilde{c}(y_1)y_1)$ is bounded away from $\bar{x}_u$ for y_1 large enough, we have in both cases that $u'(x^*(y_1, \tilde{c}(y_1)y_1))$ is bounded, thus $z(y_1) \to c$ as $y_1 \to \infty$. The continuity of L implies

$$\lim_{y_1\to\infty} x^*(y_1, \tilde{c}(y_1)y_1) = -L(c) \geq x^*(0, \mu) > \bar{x}_u.$$

By definition of L we have $-L(c) \leq \bar{x}_\ell$.

Finally, observe that $x^*(y_1, cy_1)$ is decreasing in c by (vii).

(x) v as the supremum of linear functions is convex. The second equality in (3.48) follows from the definition of x^*.

Note that $v(y_2, y_1, y_0) = y_0 g_{(y_1/y_0),(y_2/y_0)}\left(x^*\left(y_1/y_0, y_2/y_0\right)\right)$. For $y_2 \notin [u'(\bar{x}_\ell),\, u'(\bar{x}_\ell) + y_1\ell'(-\bar{x}_\ell+)]$ we have

$$\begin{aligned}\frac{\partial}{\partial y_1} g_{y_1,y_2}(x^*(y_1, y_2)) &= g'_{y_1,y_2}(x^*(y_1, y_2))\frac{\partial}{\partial y_1}x^*(y_1, y_2) - \ell(-x^*(y_1, y_2)) \\ &= -\ell(-x^*(y_1, y_2))\end{aligned}$$

and

$$\frac{\partial}{\partial y_2} g_{y_1,y_2}(x^*(y_1,y_2)) = g'_{y_1,y_2}(x^*(y_1,y_2)) \frac{\partial}{\partial y_2} x^*(y_1,y_2) - x^*(y_1,y_2)$$
$$= -x^*(y_1,y_2)$$

because $g'_{y_1,y_2}(x^*(y_1,y_2)) = 0$.

For $y_2 \in [u'(\bar{x}_\ell),\, u'(\bar{x}_\ell) + y_1 \ell'(-\bar{x}_\ell +)]$, we have $x^*(y_1,y_2) = \bar{x}_\ell$, thus

$$g(x^*(y_1,y_2)) = u(\bar{x}_\ell) - y_2 \bar{x}_\ell.$$

It follows that

$$\frac{\partial}{\partial y_1} g(x^*(y_1,y_2)) = 0 = -\ell(-\bar{x}_\ell) = -\ell(-x^*(y_1,y_2)),$$

and

$$\frac{\partial}{\partial y_2} g(x^*(y_1,y_2)) = -\bar{x}_\ell = -x^*(y_1,y_2).$$

This implies that $g(x^*(y_1,y_2))$ is continuously differentiable in $y_1 \geq 0$ and $y_2 > 0$.

Now (3.49), (3.50), and (3.51) follow from standard calculus. The last part of (x) can be shown in exactly the same way as Lemma 2.1.6(ii).

(xi) This follows from the definition of L and basic calculus, similar to the proofs of (ii) and (x). □

Proof of Lemma 3.2.4

In this section we prove Lemma 2.3.4. Note that the subjective measures Q_0 and Q_1 are assumed to be equivalent to the reference measure R. Hence, a statement holds Q_i-almost surely ($i = 0, 1$) if and only if it holds R-almost surely. We will always suppose that Assumption 3.2.1 holds, and fix a level $x_0 \in (\bar{x}_u, \infty)$ for the budget constraint. For $\lambda_1 \geq 0$, we let

$$X^*(\lambda_1) := x^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right), \tag{3.53}$$

where λ_2 is chosen such that the budget constraint $x_0 = E_P[X^*(\lambda_1)]$ is satisfied.

Lemma 3.2.13. *For each $\lambda_1 \geq 0$, the random variable $X^*(\lambda_1)$ is R-almost surely well defined.*

Proof. On $\{dP/dR = 0\} = \{dP/dQ_0 = 0\}$ we have $X^*(\lambda_1) = +\infty$. Thus, it suffices to show that $X^*(\lambda_1)$ is P-almost surely well defined.

Let $\lambda_1 \geq 0$ be fixed. The existence of a $\lambda_2 > 0$ for which $x_0 = E_P[X^*(\lambda_1)]$ follows from Lemma 3.2.12(iv) and (vi), Assumption 3.2.1(a) and monotone convergence.

Suppose now that

$$x_0 \;=\; E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right] \;=\; E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \hat{\lambda}_2\frac{dP}{dQ_0}\right)\right]. \tag{3.54}$$

for $\lambda_2 \geq \hat{\lambda}_2$. For fixed first argument, the function x^* is decreasing in its second argument by Lemma 3.2.12(v), thus

$$x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right) \leq x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \hat{\lambda}_2\frac{dP}{dQ_0}\right). \tag{3.55}$$

Because of condition (3.54), the preceding inequality (3.55) must P-almost surely be an equality. This implies that X^* is P-almost surely well defined. □

Lemma 3.2.14. *For each $\lambda_1 \geq 0$, we let $\lambda(\lambda_1)$ be the supremum of all $\lambda_2 > 0$ such that the budget constraint*

$$x_0 \;=\; E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right]$$

is satisfied. Then $\lambda(\lambda_1) \in (0,\infty)$, and the supremum is attained. Moreover, the function $\lambda(\lambda_1)/\lambda_1$ is decreasing for $\lambda_1 \in (0,\infty)$. In particular,

$$\lim_{\lambda_1\to\infty}\frac{\lambda(\lambda_1)}{\lambda_1} \in [0,\infty)$$

exists.

Proof. By Lemma 3.2.12(vi), $x^*(\lambda_1 dQ_1/dQ_0, \lambda_2 dP/dQ_0)$ converges to $\bar{x}_u$ as $\lambda_2 \to \infty$ and to infinity as $\lambda_2 \to 0$. Moreover, $E_P[x^*(\lambda_1 dQ_1/dQ_0, \lambda_2 dP/dQ_0)]$ is continuous in λ_2 by monotone convergence and Assumption 3.2.1(a). This implies the first claim, since $\bar{x}_u < x_0 < \infty$. Furthermore, $\lambda(\lambda_1)$ is indeed a maximum, since $E_P[x^*(\lambda_1 dQ_1/dQ_0, \lambda_2 dP/dQ_0)]$ is continuous in λ_2.

In order to show that $\lambda(\lambda_1)/\lambda_1$ is decreasing, let $\lambda_1' > \lambda_1 > 0$ and define $\alpha := \lambda_1'/\lambda_1 > 1$. It follows from Lemma 3.2.12(vii) that

$$\begin{aligned} x^*\left(\lambda_1'\frac{dQ_1}{dQ_0}, \lambda(\lambda_1')\frac{dP}{dQ_0}\right) &= x^*\left(\alpha\lambda_1\frac{dQ_1}{dQ_0}, \alpha\lambda_1\frac{\lambda(\lambda_1')}{\lambda_1'}\frac{dP}{dQ_0}\right) \\ &\leq x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_1\frac{\lambda(\lambda_1')}{\lambda_1'}\frac{dP}{dQ_0}\right). \end{aligned}$$

This implies that

$$x_0 = E_P\left[x^*\left(\lambda_1'\frac{dQ_1}{dQ_0}, \lambda(\lambda_1')\frac{dP}{dQ_0}\right)\right] \le E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_1\frac{\lambda(\lambda_1')}{\lambda_1'}\frac{dP}{dQ_0}\right)\right].$$

Suppose now that $\lambda(\lambda_1')/\lambda_1' > \lambda(\lambda_1)/\lambda_1$. Since x^* is decreasing in its second argument with first argument fixed, there exists $\lambda_2 \ge \lambda_1\lambda(\lambda_1')/\lambda_1' > \lambda(\lambda_1)$ such that

$$x_0 = E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right],$$

contradicting the maximality of $\lambda(\lambda_1)$. □

In order to avoid any ambiguity, we will always work with the following version of the stochastic process $(X^*(\lambda_1))_{\lambda_1\ge 0}$:

$$X^*(\lambda_1) \;:=\; x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda(\lambda_1)\frac{dP}{dQ_0}\right) \qquad (\lambda_1 \ge 0).$$

Lemma 3.2.15. *Let $\lambda_1 \ge 0$. If $(\lambda_1^{(n)})_{n\in\mathbb{N}}$ is a sequence with $\lambda_1^{(n)} \to \lambda_1$, then there exists a subsequence $(\lambda_1^{(n_j)})_{j\in\mathbb{N}}$ such that $X^*(\lambda_1^{(n_j)}) \to X^*(\lambda_1)$ R-almost surely.*

Proof. For $n \in \mathbb{N}$, we choose $\lambda_2^{(n)} = \lambda(\lambda_1^{(n)}) > 0$. In a first step we show that this sequence is both bounded and bounded away from zero.

Suppose that the sequence $(\lambda_2^{(n)})_n$ is unbounded. Then there exists an increasing subsequence $\lambda_2^{(n_j)}$ which converges to infinity as $j \to \infty$. Let $\hat\lambda := \max_{n\in\mathbb{N}} \lambda_1^{(n)}$. By Lemma 3.2.12(v)&(vi),

$$X^*\left(\lambda_1^{(n_j)}\right) \;\le\; x^*\left(\hat\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2^{(n_j)}\frac{dP}{dQ_0}\right) \overset{j\to\infty}{\searrow} \;\bar{x}_u.$$

Due to Assumption 3.2.1(a), the monotone convergence theorem and the definition of $X^*(\lambda_1)$ imply that $x_0 \le \bar{x}_u$, a contradiction. Thus, $(\lambda_2^{(n)})_n$ is bounded.

Suppose now that zero is an accumulation point of $(\lambda_2^{(n)})_n$. Then there exists a decreasing subsequence $\lambda_2^{(n_j)}$ which converges to zero as $j \to \infty$. Let $\hat\lambda := \min_{n\in\mathbb{N}} \lambda_1^{(n)}$. By Lemma 3.2.12(v)&(vi),

$$X^*\left(\lambda_1^{(n_j)}\right) \;\ge\; x^*\left(\hat\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2^{(n_j)}\frac{dP}{dQ_0}\right) \overset{j\to\infty}{\nearrow} \;\infty.$$

The monotone convergence theorem and the definition of $X^*(\lambda_1)$ imply that $x_0 = \infty$, a contradiction. Thus, $(\lambda_2^{(n)})_n$ is bounded away from zero.

For any convergent sequence $(\lambda_1^{(n)})_n$ with limit λ_1, we can now find a subsequence $(\lambda_1^{(n_j)})$ such that $(\lambda_2^{(n_j)})$ is convergent with limit, say, $\lambda_2 \in (0,\infty)$. Hence we have $\lim_{j\to\infty} X^*(\lambda_1^{(n_j)}) = x^*(\lambda_1 dQ_1/dQ_0, \lambda_2 dP/dQ_0)$ R-almost surely on $\{dP/dQ_0 > 0\}$. But on $\{dP/dQ_0 = 0\}$ we have $X^*(\lambda_1^{(n_j)}) = \infty = x^*(\lambda_1 dQ_1/dQ_0, \lambda_2 dP/dQ_0)$. Thus we obtain R-almost sure convergence on Ω.

Furthermore, choosing $\tilde{\lambda}_1 := \min_{j\in\mathbb{N}} \lambda_1^{(n_j)}, \hat{\lambda}_1 := \max_{j\in\mathbb{N}} \lambda_1^{(n_j)} \in [0,\infty)$, and $\tilde{\lambda}_2 := \max_{j\in\mathbb{N}} \lambda_2^{(n_j)}, \hat{\lambda}_2 := \min_{j\in\mathbb{N}} \lambda_2^{(n_j)} \in (0,\infty)$, we have the bounds

$$x^*\left(\tilde{\lambda}_1 \frac{dQ_1}{dQ_0}, \tilde{\lambda}_2 \frac{dP}{dQ_0}\right) \;\leq\; X^*(\lambda_1^{(n_j)}) \;\leq\; x^*\left(\hat{\lambda}_1 \frac{dQ_1}{dQ_0}, \hat{\lambda}_2 \frac{dP}{dQ_0}\right).$$

By Lebesgue's dominated convergence theorem and Assumption 3.2.1(a) we obtain therefore,

$$x_0 = \lim_{j\to\infty} E_P\left[X^*\left(\lambda_1^{(n_j)}\right)\right] = E_P\left[x^*\left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0}\right)\right].$$

By Lemma 3.2.13, this implies $X^*(\lambda_1) = x^*\left(\lambda_1 dQ_1/dQ_0, \lambda_2 dP/dQ_0\right)$. □

For the proof of the main result we will need to investigate the function

$$k : \lambda_1 \mapsto E_{Q_1}\left[\ell\left(-X^*\left(\lambda_1\right)\right)\right].$$

Lemma 3.2.16. *The function k is continuous.*

Proof. Let $(\lambda_1^{(n)})_n$ be a sequence of non-negative reals converging to λ_1. We need to show that any accumulation point k^* of $(k(\lambda_1^{(n)}))_n$ is equal to $k(\lambda_1)$. By Lemma 3.2.15 we can choose a subsequence $(\lambda_1^{(n_j)})$ such that both $k(\lambda_1^{(n_j)}) \to k^*$ and $X^*(\lambda_1^{(n_j)}) \to X^*(\lambda_1)$ R-almost surely. We have

$$\begin{aligned} k^* &= \lim_{j\to\infty} k\left(\lambda_1^{(n_j)}\right) \\ &= \lim_{j\to\infty} E_{Q_1}\left[\ell\left(-X^*\left(\lambda_1^{(n_j)}\right)\right)\right] \\ &\overset{(*)}{=} E_{Q_1}\left[\ell(-X^*(\lambda_1))\right] = k(\lambda_1). \end{aligned}$$

Equality $(*)$ follows from the dominated convergence theorem, since for all $j \in \mathbb{N}$ we have the inequality

$$0 \;\leq\; \ell\left(-X^*\left(\lambda_1^{(n_j)}\right)\right) \;\leq\; \ell\left(-x^*\left(\hat{\lambda}_1 \frac{dQ_1}{dQ_0}, \hat{\lambda}_2 \frac{dP}{dQ_0}\right)\right) \tag{3.56}$$

with $\hat{\lambda}_1 = \min_j \lambda_1^{(n_j)}$, $\hat{\lambda}_2 = \max_j \lambda_2^{(n_j)}$. The upper bound in (3.56) is Q_1-integrable by Assumption 3.2.1(b). □

Recall that L is the generalized inverse of the derivative of the loss function ℓ, see equation (3.22). L is a continuous function which is strictly increasing on $[\ell'(-\bar{x}_\ell+), \ell'(-\bar{x}_u)]$. With this function we can characterize the asymptotic behavior of $X^*(\lambda_1)$, $\ell(-X^*(\lambda_1))$, and of the expectations of these quantities for $\lambda_1 \to \infty$.

Lemma 3.2.17. *Let* $c_{P,Q_1} := \lim_{\lambda_1\to\infty} \lambda(\lambda_1)/\lambda_1$.

(i) Suppose $x_0 > \bar{x}_\ell$. *In this case, we have* $c_{P,Q_1} = 0$ *and* $\lim_{\lambda_1\to\infty} k(\lambda_1) = 0$.

(ii) Suppose $\bar{x}_u < x_0 < \bar{x}_\ell$.
In this case, we have $c_{P,Q_1} > 0$ *and*

$$\lim_{\lambda_1\to\infty} X^*(\lambda_1) = -L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right) \qquad P-\textit{almost surely.}$$

Furthermore, c_{P,Q_1} *is a solution to the equation*

$$x_0 = -E_P\left[L\left(c\frac{dP}{dQ_1}\right)\right], \tag{3.57}$$

and

$$\lim_{\lambda_1\to\infty} k(\lambda_1) = E_{Q_1}\left[\ell\left(L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right)\right)\right]$$

(iii) If $x_0 = \bar{x}_\ell$, *then* $\lim_{\lambda_1\to\infty} k(\lambda_1) = 0$.

Proof. (o) Let $\tilde{c} : \mathbb{R}_+ \to \mathbb{R}_+$ be decreasing with $\lim_{y_1\to\infty} \tilde{c}(y_1) = c > 0$. We will repeatedly use the facts that

$$x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_1\tilde{c}(\lambda_1)\frac{dP}{dQ_0}\right) \overset{\lambda_1\to\infty}{\longrightarrow} -L\left(c\frac{dP}{dQ_1}\right) \qquad P-\text{almost surely} \tag{3.58}$$

and

$$\ell\left(-x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_1\tilde{c}(\lambda_1)\frac{dP}{dQ_0}\right)\right) \overset{\lambda_1\to\infty}{\longrightarrow} \ell\left(L\left(c\frac{dP}{dQ_1}\right)\right) \quad R-\text{almost surely.} \tag{3.59}$$

If $\tilde{c}(\lambda_1) \equiv c$, then the convergence is monotone by Lemma 3.2.12(vii) and

$$E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_1 c\frac{dP}{dQ_0}\right)\right] \overset{\lambda_1\to\infty}{\searrow} -E_P\left[L\left(c\frac{dP}{dQ_1}\right)\right] \tag{3.60}$$

and

$$E_{Q_1}\left[\ell\left(-x^*\left(\lambda_1\frac{dQ_1}{dQ_0},\lambda_1 c\frac{dP}{dQ_0}\right)\right)\right] \overset{\lambda_1\to\infty}{\nearrow} E_{Q_1}\left[\ell\left(L\left(c\frac{dP}{dQ_1}\right)\right)\right]. \qquad (3.61)$$

The statements (3.58)-(3.61) follow from Lemma 3.2.12(ix) in the following way: Since $Q_1 \sim Q_0 \sim R$, we have R-almost surely

$$x^*\left(\lambda_1\frac{dQ_1}{dQ_0},\lambda_1\tilde{c}(\lambda_1)\frac{dP}{dQ_0}\right) = x^*\left(\lambda_1\frac{dQ_1}{dQ_0},\lambda_1\tilde{c}(\lambda_1)\frac{dQ_1}{dQ_0}\frac{dP}{dQ_1}\right).$$

This expression converges to $-L\left(cdP/dQ_1\right)$ R-almost surely on $\{dP/dQ_1 > 0\}$ due to Lemma 3.2.12(ix), which implies (3.58).
Furthermore, on $\{dP/dQ_1 = 0\}$ we have

$$x^*\left(\lambda_1\frac{dQ_1}{dQ_0},\lambda_1\tilde{c}(\lambda_1)\frac{dQ_1}{dQ_0}\frac{dP}{dQ_1}\right) = \infty$$

and $-L\left(cdP/dQ_1\right) = \bar{x}_\ell$, hence

$$\ell\left(-x^*\left(\lambda_1\frac{dQ_1}{dQ_0},\lambda_1\tilde{c}(\lambda_1)\frac{dP}{dQ_0}\right)\right) = 0 = \ell\left(L\left(c\frac{dP}{dQ_0}\right)\right),$$

and (3.59) follows. The properties (3.60) and (3.61) follow now from Lemma 3.2.12(ix), Assumption 3.2.1(a)&(b), and the monotone convergence theorem.

We will now prove part (i). Let $x_0 > \bar{x}_\ell$.

(i-a) Suppose $c_{P,Q_1} > 0$. By Lemma 3.2.14 we have $\lambda(\lambda_1)/\lambda_1 \geq c_{P,Q_1} > 0$ for $\lambda_1 > 0$. Thus by Lemma 3.2.12(v)&(vii), for $\lambda_1 \geq \lambda_1' > 0$:

$$X^*(\lambda_1) \leq x^*\left(\lambda_1\frac{dQ_1}{dQ_0},\ c_{P,Q_1}\cdot\lambda_1\frac{dP}{dQ_0}\right) \leq x^*\left(\lambda_1'\frac{dQ_1}{dQ_0},\ c_{P,Q_1}\cdot\lambda_1'\frac{dP}{dQ_0}\right).$$

From (3.60) we obtain

$$\begin{aligned} x_0 = E_P[X^*(\lambda_1)] &\leq E_P\left[x^*\left(\lambda_1'\frac{dQ_1}{dQ_0},\ c_{P,Q_1}\cdot\lambda_1'\frac{dP}{dQ_0}\right)\right] \\ &\overset{\lambda_1'\to\infty}{\searrow} E_P\left[-L\left(c_{P,Q_1}\cdot\frac{dP}{dQ_1}\right)\right] \leq \bar{x}_\ell, \end{aligned}$$

a contradiction. Thus, $c_{P,Q_1} = 0$.

(i-b) Since $c_{P,Q_1} = 0$, it follows from Lemma 3.2.12(v) that for any $\epsilon > 0$ and λ_1 large enough, $X^*(\lambda_1) \geq x^*\left(\lambda_1 dQ_1/dQ_0, \epsilon\lambda_1 dP/dQ_0\right)$. With $k(\lambda_1) = E_{Q_1}[\ell(-X^*(\lambda_1))]$, this implies

$$\begin{aligned} 0 \leq \liminf_{\lambda_1\to\infty} k(\lambda_1) &\leq \limsup_{\lambda_1\to\infty} k(\lambda_1) \\ &\leq \lim_{\lambda_1\to\infty} E_{Q_1}\left[\ell\left(-x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \epsilon\lambda_1\frac{dP}{dQ_0}\right)\right)\right] \\ &= E_{Q_1}\left[\ell\left(L\left(\epsilon\frac{dP}{dQ_1}\right)\right)\right] \end{aligned}$$

due to (3.61). Furthermore the dominated convergence theorem and Assumption 3.2.1(e) imply

$$\lim_{\epsilon\to 0} E_{Q_1}\left[\ell\left(L\left(\epsilon\frac{dP}{dQ_1}\right)\right)\right] = 0,$$

since ℓ and L are increasing. Thus, $\lim_{\lambda_1\to\infty} k(\lambda_1) = 0$.

We will now prove part (ii). Let $x_0 < \bar{x}_\ell$.

(ii-a) Let us first show that $x_0 < \bar{x}_\ell$ implies $c_{P,Q_1} > 0$. Suppose $c_{P,Q_1} = 0$. Then for every $\epsilon > 0$ there exists $\lambda_1' > 0$ such that for $\lambda_1 \geq \lambda_1'$

$$X^*(\lambda_1) \geq x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_1\epsilon\frac{dP}{dQ_0}\right). \tag{3.62}$$

From (3.60) we obtain for $\lambda_1 \geq \lambda_1'$

$$x_0 = E_P[X^*(\lambda_1)] \geq E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_1\epsilon\frac{dP}{dQ_0}\right)\right] \overset{\lambda_1\to\infty}{\searrow} E_P\left[-L\left(\epsilon\frac{dP}{dQ_1}\right)\right].$$

Thus, the monotone convergence theorem and Assumption 3.2.1(d) imply

$$x_0 \geq \lim_{\epsilon\to 0} E_P\left[-L\left(\epsilon\frac{dP}{dQ_1}\right)\right] = \bar{x}_\ell,$$

a contradiction. This implies $c_{P,Q_1} > 0$. The first result now follows from (3.58).

(ii-b) We will now show that $c_{P,Q_1} = \lim_{\lambda_1\to\infty}\lambda(\lambda_1)/\lambda_1$ is a solution of Equation (3.57). It is not difficult to see that for $\lambda_1 > n \in \mathbb{N}$ we have

$$\begin{aligned} -L\left(\frac{\lambda(n)}{n}\cdot\frac{dP}{dQ_1}\right) &\overset{(1)}{\leq} x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \frac{\lambda(n)}{n}\cdot\lambda_1\cdot\frac{dP}{dQ_0}\right) \overset{(2)}{\leq} X^*(\lambda_1) \\ &\overset{(3)}{\leq} x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, c\lambda_1\frac{dP}{dQ_0}\right) \overset{(4)}{\leq} x^*\left(\frac{dQ_1}{dQ_0}, c\frac{dP}{dQ_0}\right). \end{aligned} \tag{3.63}$$

Inequality (1) follows from Lemma 3.2.12(ix). Inequalities (2) and (3) follow from Lemma 3.2.12(v) and the fact that $(\lambda(n)/n) \cdot \lambda_1 \geq \lambda(\lambda_1) \geq c\lambda_1$ for $\lambda_1 > n$, since $\lambda(\lambda_1)/\lambda_1$ decreases to c as $\lambda_1 \to \infty$. Inequality (4) follows from Lemma 3.2.12 (vii).

Due to (3.58) and Assumption 3.2.1(a)&(d), we may apply the dominated convergence theorem to obtain

$$x_0 = E_P\left[X^*(\lambda_1)\right] \to E_P\left[-L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right)\right].$$

Thus, c_{P,Q_1} is a solution to equation (3.57).

Analogously, due to (3.59) and Assumption 3.2.1(b)&(e) we may apply the dominated convergence theorem to obtain

$$\lim_{\lambda_1\to\infty} k(\lambda_1) = \lim_{\lambda_1\to\infty} E_{Q_1}\left[\ell\left(-X^*(\lambda_1)\right)\right] = E_{Q_1}\left[\ell\left(L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right)\right)\right].$$

(iii) Let $x_0 = \bar{x}_\ell$. If $c_{P,Q_1} = 0$, argue as in part (i-b) to verify the claim. If $c_{P,Q_1} > 0$, argue as in part (ii-b) to show that $x_0 = E_P\left[-L\left(c_{P,Q_1}dP/dQ_1\right)\right]$. Since $-L\left(c_{P,Q_1}dP/dQ_1\right) \leq \bar{x}_\ell = x_0$, this implies $-L\left(c_{P,Q_1}dP/dQ_1\right) = \bar{x}_\ell$ P-almost surely and hence Q_1-almost surely on $\{dP/dQ_1 > 0\}$. But since $-L(0) = \bar{x}_\ell$, it holds Q_1-almost surely on Ω.

Analogously to part (ii-b), we obtain finally

$$\lim_{\lambda_1\to\infty} k(\lambda_1) = E_{Q_1}\left[\ell\left(L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right)\right)\right] = E_{Q_1}\left[\ell(-\bar{x}_\ell)\right] = 0.$$

□

We summarize the asymptotic behavior of k in the following corollary.

Corollary 3.2.18. *Suppose that Assumption 3.2.1 holds, and let* $x_0 > \bar{x}_u$. *Let* $\tilde{\lambda}_2$ *be the unique solution to the equation*

$$x_0 \;=\; E_P\left[I\left(\tilde{\lambda}_2\frac{dP}{dQ_0}\right)\right].$$

The asymptotic behavior of k *can be characterized in the following way.*

$$\begin{aligned}
\lim_{\lambda_1\to 0} k(\lambda_1) &= E_{Q_1}\left[\ell\left(-I\left(\tilde{\lambda}_2\frac{dP}{dQ_1}\right)\right)\right],\\
\lim_{\lambda_1\to\infty} k(\lambda_1) &= \begin{cases} 0 & \text{if } x_0 \geq \bar{x}_\ell,\\ E_{Q_1}\left[\ell\left(L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right)\right)\right] & \text{if } x_0 < \bar{x}_\ell,\end{cases}
\end{aligned}$$

where c_{P,Q_1} *is a solution of Equation* (3.57).

Proof. Note that $X^*(0) = I(\tilde{\lambda}_2 dP/dQ_0)$. Hence the first claim follows from Lemma 3.2.16. The second one is only a reformulation of Lemma 3.2.17. $\square$

Finally, we arrive at the following conclusion, which finishes the proof of Lemma 3.2.4.

Corollary 3.2.19. *Suppose that Assumption 3.2.1 holds and let $x_0 > \bar{x}_u$. By $\mathcal{R}(k)$ we denote the range of k. It holds $(a, b] \subseteq \mathcal{R}(k)$ with*

$$a = \begin{cases} 0 & \text{if } x_0 \geq \bar{x}_\ell, \\ E_{Q_1}\left[\ell\left(L\left(c_{P,Q_1}\frac{dP}{dQ_1}\right)\right)\right], & \text{if } x_0 < \bar{x}_\ell, \end{cases}$$

$$b = E_{Q_1}\left[\ell\left(-I\left(\tilde{\lambda}_2 \frac{dP}{dQ_0}\right)\right)\right],$$

where $\tilde{\lambda}_2$ and c_{P,Q_1} are chosen as in Corollary 3.2.18.

Proof. The proof is immediate from Lemma 3.2.16 and Corollary 3.2.18. $\square$

3.3 The Robust Problem in an Incomplete Market

In this section we solve the robust utility maximization problem (3.18) under both a budget and a risk constraint. We consider utility functions that are defined on the positive halfline, i.e., with $\bar{x}_u = 0$. In order to keep the presentation clear, we postpone all proofs of auxiliary results to Section 3.3.1. Similar to the procedure in Chapter 2, we first show the convexity of certain functions in Lemma 3.3.3, which we will later use to obtain the existence of our Lagrange multipliers. Then we solve the robust loss minimization problem. Here the proofs are very similar to the ones of the robust utility maximization problem without risk constraint in Chapter 2. Then we show the existence of a pair of Lagrange multipliers $(\lambda_1^*, \lambda_2^*)$ that minimizes a certain convex function. This gives us the minimization problem for determining the worst case measures P^*, Q_1^*, and Q_0^* in Proposition 3.3.9. These measures are characterized in Proposition 3.3.12. Finally, we state the solution to the robust utility maximization problem under a joint budget and risk constraint in Theorem 3.3.13.

Recall the definitions of the generalized divergences $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0)$ and $\tilde{v}_c(P|Q_1)$ from (3.33) and (3.34). In the robust case we replace Assumption 3.2.7 by the following robust version:

Assumption 3.3.1.

$$\inf_{P\in\mathcal{P}^T}\inf_{Q_1\in\mathcal{Q}_1}\inf_{Q_0\in\mathcal{Q}_0} v_{0,1}(P|Q_1|Q_0)<\infty. \tag{3.64}$$

Note that, since $\bar{x}_u=0$, we have $\tilde{v}_c(P|Q_1)\le 0$ for all $P\in\mathcal{P}^T$ and $Q_1\in\mathcal{Q}_1$ by Remark 3.2.6(ii). Hence the robust version of Assumption 3.2.7(ii) is automatically satisfied. Furthermore (3.64) is in fact equivalent to the assumption $\inf_{P\in\mathcal{P}^T}\inf_{Q_0\in\mathcal{Q}_0} v(P|Q_0)<\infty$ from Chapter 2 since $v_{0,1}(P|Q_1|Q_0)=v(P|Q_0)$, where the right-hand side is the v-divergence defined in (2.19).

Remark 3.3.2. *Due to the assumption* (2.9) *of reasonable asymptotic elasticity on the utility function and due to Remark 3.2.8(i), Assumption 3.3.1 is equivalent to*

$$\inf_{P\in\mathcal{P}^T}\inf_{Q_1\in\mathcal{Q}_1}\inf_{Q_0\in\mathcal{Q}_0} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0)<\infty \quad \textit{for all } \lambda_1\ge 0,\ \lambda_2>0.$$

Let us define the subset $\mathcal{C}_f$ of measures $P\in\mathcal{P}^T$, $Q_1\in\mathcal{Q}_1$, and $Q_0\in\mathcal{Q}_0$ with finite generalized divergence, i.e.,

$$\mathcal{C}_f:=\left\{(P,Q_1,Q_0):\ v_{0,1}(P|Q_1|Q_0)<\infty\right\}. \tag{3.65}$$

Note that in fact this does not restrict the measures in $\mathcal{Q}_1$ since $v_{0,1}(P|Q_1|Q_0)$ is independent of Q_1.

As in Chapter 2, we have to show the existence of a pair of suitable Lagrange multipliers such that the constraints are satisfied. Since we again have to solve two problems, first the one of loss minimization and then the one of utility maximization, we have to deal with the following four convex functions.

Lemma 3.3.3. *Let Assumption 3.3.1 hold. The function*

$$H(\lambda_1,\lambda_2):=\inf_{P\in\mathcal{P}^T}\inf_{Q_1\in\mathcal{Q}_1}\inf_{Q_0\in\mathcal{Q}_0} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0)+\lambda_1x_1+\lambda_2x_0$$

is convex on $[0,\infty)\times(0,\infty)$. *Furthermore, the function*

$$\tilde{H}(c)=\inf_{P\in\mathcal{P}^T}\inf_{Q_1\in\mathcal{Q}_1}\tilde{v}_c(P|Q_1)+cx_0$$

is convex on $(0,\infty)$, *and the functions*

$$G_{P,Q_1,Q_0}(\lambda_1):=\inf_{\lambda_2>0}\left\{v_{\lambda_1,\lambda_2}(P|Q_1|Q_0)+\lambda_1x_1+\lambda_2x_0\right\}$$

for $(P,Q_1,Q_0)\in\mathcal{C}_f$, *and*

$$G(\lambda_1):=\inf_{P\in\mathcal{P}^T}\inf_{Q_1\in\mathcal{Q}_1}\inf_{Q_0\in\mathcal{Q}_0} G_{P,Q_1,Q_0}(\lambda_1)$$

are convex on $[0,\infty)$.

Proof. See Section 3.3.1 below. □

As in the previous section, let us first solve the problem of minimizing the expected loss over all contingent claims $Y \geq 0$ under the budget constraint (3.4) of an incomplete market, i.e.,

$$\begin{aligned} &\text{Minimize } \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-Y)] \text{ over all } Y \geq 0 \\ &\text{with } Y \in L^1(P) \text{ for all } P \in \mathcal{P}^T \text{ and } \sup_{P \in \mathcal{P}^T} E_P[Y] \leq x_0. \end{aligned} \tag{3.66}$$

In order to solve this problem, we need the following auxiliary result, which basically coincides with Lemma 2.3.1.

Lemma 3.3.4. *Let Assumption 3.3.1 hold and let $x_0 \in (0, \bar{x}_\ell)$. There exists $c^* \in (0, \infty)$ that minimizes the convex function*

$$\tilde{H}(c) = \inf_{P \in \mathcal{P}^T} \inf_{Q_1 \in \mathcal{Q}_1} \tilde{v}_c(P|Q_1) + cx_0.$$

Proof. Using the convexity of $\tilde{H}$ shown in Lemma 3.3.3 and replacing I by $-L$, this proof follows exactly the lines of the one of Lemma 2.3.1. □

Let such a minimizer c^* be fixed. The dual problem of the loss minimization problem consists of minimizing $\tilde{v}_{c^*}(P|Q_1)$ over $\mathcal{P}^T$ and $\mathcal{Q}_1$.

Proposition 3.3.5. *There exist $\tilde{P} \in \mathcal{P}^T$ and $\tilde{Q}_1 \in \mathcal{Q}_1$ that achieve the infimum of $\tilde{v}_{c^*}(P|Q_1)$ over the sets $\mathcal{P}^T$ and $\mathcal{Q}_1$.*

Proof. This follows from Theorem 1.2.8, where only the convexity of the function $\tilde{v}$ and the fact $\lim_{c\to\infty} \tilde{v}(c,1)/c = 0$ is needed. □

The following proposition gives the solution to the risk minimization problem (3.66). The procedure for obtaining this solution is the same as for the robust utility maximization problem without risk constraint in Chapter 2: One uses the result of Proposition 3.2.2 without model uncertainty and a characterization of the minimizing measures $\tilde{P}$ and $\tilde{Q}_1$.

Proposition 3.3.6. *For $x_0 \in (0, \bar{x}_\ell)$, the solution to Problem* (3.66) *is R-almost surely unique on the set $\{d\tilde{P}/dR > 0\}$ and given by*

$$Y^* = -L\left(c^* \frac{d\tilde{P}}{d\tilde{Q}_1}\right).$$

Furthermore, Problem (3.66) *is equivalent to the classical problem* (3.21) *under the measures $\tilde{P}$ and $\tilde{Q}_1$, and*

$$-\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-Y^*)] = -E_{\tilde{Q}_1}[\ell(-Y^*)] = \tilde{v}_{c^*}(\tilde{P}|\tilde{Q}_1) + c^* x_0. \tag{3.67}$$

Proof. Existence and characterization of the solution follow from Propositions 3.2.2 and 3.3.5 in the same way as in Theorem 2.3.10: Inequality (3.25) applied to $\tilde{P}$ and $\tilde{Q}_1$ implies

$$
\begin{aligned}
-\sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-Y)] &\le -E_{\tilde{Q}_1}[\ell(-Y)] \\
&\le \tilde{v}_{c^*}(\tilde{P}|\tilde{Q}_1) + c^* x_0 \qquad (3.68)\\
&= -E_{\tilde{Q}_1}\left[\ell\left(-Y^*\right)\right] + c^*\left(x_0 - E_{\tilde{P}}\left[Y^*\right]\right)
\end{aligned}
$$

for any $Y \ge 0$ satisfying the budget constraint. Note that the convex function $c \mapsto \tilde{v}_c(\tilde{P}|\tilde{Q}_1) + cx_0$ attains its minimum in c^*. Thus Corollary 3.2.11(ii) implies $x_0 = E_{\tilde{P}}[Y^*]$. Furthermore, Proposition 2.3.8 applied to the utility function $\tilde{u} := -\ell(-\cdot)$ with $\tilde{I} := -L$ leads to $\sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-Y^*)] = E_{\tilde{Q}_1}[\ell(-Y^*)]$ and $\sup_{P\in\mathcal{P}^T} E_P[Y^*] = E_{\tilde{P}}[Y^*]$. Note that here we do not need any further condition on the set $\mathcal{Q}_1$ such as Assumption 2.3.2 since $\tilde{v}_c(P|Q_1) \le 0$. Thus, Inequality (3.68) implies $-\sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-Y)] \le -\sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-Y^*)]$, and Y^* satisfies the budget constraint. This concludes the proof of the optimality of Y^* and of (3.67).

Y^* is the solution to the classical loss minimization problem (3.21) under the measures $\tilde{Q}_1$ and $\tilde{P}$. In order to show uniqueness, assume that $\tilde{Y}$ solves Problem (3.66). Then we have $E_{\tilde{P}}[\tilde{Y}] \le x_0$ and hence

$$
\sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-\tilde{Y})] \ge E_{\tilde{Q}_1}[\ell(-\tilde{Y})] \ge E_{\tilde{Q}_1}[\ell(-Y^*)].
$$

The second inequality holds strictly unless $\tilde{Y} = Y^*$ R-almost surely on $\{d\tilde{P}/dR > 0\}$. This follows from the fact that Y^* is the solution to Problem (3.21) under $\tilde{P}$ and $\tilde{Q}_1$ and from the uniqueness result in Proposition 3.2.2. But the strict inequality is a contradiction to $E_{\tilde{Q}_1}[\ell(-Y^*)] = \sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-Y^*)]$. Thus $\tilde{Y} = Y^*$ R-almost surely on $\{d\tilde{P}/dR > 0\}$. □

Remark 3.3.7. *If $\tilde{v}(c,1)$ is not strictly convex in c, then c^* and the density $d\tilde{P}/d\tilde{Q}_1$ are not necessarily unique. The proposition shows that even in this case, Y^* is still unique at least $\tilde{P}$-almost surely.*

Let us now consider the robust utility maximization problem (3.18) under a joint budget and risk constraint. We first need to show the existence of a pair of suitable Lagrange multipliers, similar to Lemma 2.3.4. Recall the definition of the set $\mathcal{C}_f$ of measures with finite generalized divergence from (3.65). As in the proof of Lemma 3.2.4 in Section 3.2.2, we define for $\lambda_1 \ge 0$ and $(P, Q_1, Q_0) \in \mathcal{C}_f$ the constant $\lambda(\lambda_1, P, Q_1, Q_0)$ as the maximal value

$\lambda_2 > 0$ such that

$$x_0 \;=\; E_P\left[x^*\left(\lambda_1\frac{dQ_1}{dQ_0}, \lambda_2\frac{dP}{dQ_0}\right)\right].$$

Lemma 3.3.8. *Let x_1, $x_0 > 0$, and let Assumption 3.3.1 hold, that is, assume that* $\inf_{P\in\mathcal{P}^T}\inf_{Q_1\in\mathcal{Q}_1}\inf_{Q_0\in\mathcal{Q}_0} v_{0,1}(P|Q_1|Q_0) < \infty$*. If*

$$x_1 > \lim_{\lambda_1\to\infty}\sup_{(P,Q_1,Q_0)\in\mathcal{C}_f} E_{Q_1}\left[\ell\left(L\left(\frac{\lambda(\lambda_1,P,Q_1,Q_0)}{\lambda_1}\frac{dP}{dQ_1}\right)\right)\right],$$

then there exists a minimizer $(\lambda_1^*, \lambda_2^*) \in [0,\infty)\times(0,\infty)$ of the convex function

$$\inf_{P\in\mathcal{P}^T}\inf_{Q_1\in\mathcal{Q}_1}\inf_{Q_0\in\mathcal{Q}_0} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0.$$

Proof. See Section 3.3.1 below. □

The following proposition will give us our worst case measures. The proof is similar to the one of Theorem 1.2.8, where we showed the existence of a pair of measures that minimizes the f-divergence.

Proposition 3.3.9. *Assume that the sets $\mathcal{Q}_0$ and $\mathcal{Q}_1$ satisfy the compactness assumptions 3.1.1 and 3.1.3. Then there exist measures $P^* \in \mathcal{P}^T$, $Q_1^* \in \mathcal{Q}_1$, and $Q_0^* \in \mathcal{Q}_0$ that achieve the infimum of the convex functional $v_{\lambda_1^*,\lambda_2^*}(P|Q_1|Q_0)$ over the sets $\mathcal{P}^T$, $\mathcal{Q}_1$, and $\mathcal{Q}_0$.*

Proof. See Section 3.3.1 below. □

We need one more assumption on our set $\mathcal{Q}_0$. As Assumption 2.3.2 in the previous chapter, it will allow us to characterize Q_0^* as the worst case measure for the utility evaluation of the optimal contingent claim.

Assumption 3.3.10. *For any $Q_0 \in \mathcal{Q}_0$, there is $\alpha \in (0,1]$ such that $v_{\lambda_1^*,\lambda_2^*}(P^*|Q_1^*|\alpha Q_0 + (1-\alpha)Q_0^*) < \infty$.*

Remark 3.3.11. *If $u(\infty) < \infty$, then Assumption 3.3.10 is automatically satisfied. Indeed, let $Q_0 \in \mathcal{Q}_0$, $\alpha \in (0,1)$, and define $\psi_0^* := dQ_0^*/dR$, $\psi_0 := dQ_0/dR$, $\psi_0^\alpha := \alpha\psi_0 + (1-\alpha)\psi_0^*$, $\psi_1^* := dQ_1^*/dR$, and $\phi^* := dP^*/dR$. The convex function $f(\psi_0) := v(\lambda_2^*\phi^*, \lambda_1^*\psi_1^*, \psi_0)$ has increasing derivative $f'(\psi_0) = u(x^*(\lambda_1^*\psi_1^*/\psi_0, \lambda_2^*\phi^*/\psi_0)) \le u(\infty)$ due to Lemma 3.2.12(vii)&(x). Hence*

$$\begin{aligned} f(\psi_0^\alpha) &\le f(\psi_0^*) - f'(\psi_0^\alpha)(\psi_0^* - \psi_0^\alpha) \\ &\le f(\psi_0^*) - f'((1-\alpha)\psi_0^*)\psi_0^* + u(\infty)\psi_0^\alpha \\ &= f(\psi_0^*) - u\left(x^*\left(\frac{\lambda_1^*}{1-\alpha}\frac{\psi_1^*}{\psi_0^*}, \frac{\lambda_2^*}{1-\alpha}\frac{\phi^*}{\psi_0^*}\right)\right)\psi_0^* + u(\infty)\psi_0^\alpha, \end{aligned}$$

which is in $L^1(R)$ due to Assumption 3.3.1 and Lemma 3.2.9(i).

The following proposition together with Theorem 3.3.13 below shows that P^*, Q_1^* and Q_0^* are worst case measures for pricing, for the loss evaluation, and for the utility evaluation of the optimal contingent claim, respectively, if both constraints are binding. It is the analogue result to Proposition 2.3.8 for the utility maximization problem under a joint budget and risk constraint. As the proposition in Chapter 2, it goes back to Theorem 5 by Rüschendorf [1984].

Proposition 3.3.12. *Let Assumptions 3.3.1 and 3.3.10 hold and define*

$$X^* := x^* \left(\lambda_1^* \frac{dQ_1^*}{dQ_0^*}, \lambda_2^* \frac{dP^*}{dQ_0^*} \right).$$

Then

(i) $X^* \in L^1(P)$ *for all* $P \in \mathcal{P}^T$ *and*

$$E_{P^*}[X^*] = \sup_{P \in \mathcal{P}^T} E_P[X^*], \tag{3.69}$$

(ii) $\ell(-X^*) \in L^1(Q_1)$ *for all* $Q_1 \in \mathcal{Q}_1$ *and*

$$E_{Q_1^*}[\ell(-X^*)] = \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-X^*)], \tag{3.70}$$

(iii) $u(X^*) \in L^1(Q_0)$ *for all* $Q_0 \in \mathcal{Q}_0$ *and*

$$E_{Q_0^*}[u(X^*)] = \inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X^*)]. \tag{3.71}$$

Proof. See Section 3.3.1 below. □

Finally, we are able to state the main result of this chapter, which gives the solution to the robust utility maximization problem (3.18) under both a budget and a risk constraint. Recall that $v_{0,\lambda_2}(P|Q_1|Q_0)$ does not depend on Q_1. Uniqueness in the following is meant in the R-almost sure sense.

Theorem 3.3.13. *Let the sets $\mathcal{Q}_0$ and $\mathcal{Q}_1$ satisfy the compactness assumptions 3.1.1 and 3.1.3, let the integrability assumptions 3.3.1 and 3.3.10 hold, and let x_1, $x_0 > 0$. Define Y^* as the loss-minimizing claim from Proposition 3.3.6. Furthermore, let $\tilde{\lambda}_2$ be a minimizer of the convex function*

$$\inf_{P \in \mathcal{P}^T} \inf_{Q_0 \in \mathcal{Q}_0} v_{0,\lambda_2}(P|Q_1|Q_0) + \lambda_2 x_0,$$

and $\hat{P}$ and $\hat{Q}_0$ minimizer of $v_{0,\tilde{\lambda}_2}(P|Q_1|Q_0)$ over $\mathcal{P}^T$ and $\mathcal{Q}_0$.

(i) *If* $x_0 < \bar{x}_\ell$ *and* $x_1 < \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}\left[\ell\left(-Y^*\right)\right]$, *then there is no contingent claim which satisfies both constraints.*

(ii) *Assume that* $x_0 < \bar{x}_\ell$ *and* $x_1 = \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}\left[\ell\left(-Y^*\right)\right]$.

If $u\left(Y^*\right)^- \in L^1(Q_0)$ *for all* $Q_0 \in \mathcal{Q}_0$, *then*

$$X^* := Y^* \cdot 1_{\left\{\frac{d\tilde{P}}{dR} > 0\right\}} + \infty \cdot 1_{\left\{\frac{d\tilde{P}}{dR} = 0\right\}}$$

is a solution to the maximization problem (3.18), *and both constraints are binding. Otherwise the maximization problem has no solution.* X^* *is the unique solution on the set* $\{d\tilde{P}/dR > 0\}$.

(iii) *Assume that* $\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0))] < x_1$.

Then

$$X^* := I\left(\tilde{\lambda}_2 \frac{d\hat{P}}{d\hat{Q}_0}\right)$$

is the unique solution to the maximization problem (3.18), *and the UBSR constraint is not binding.*

(iv) *Let* $\lambda(\lambda_1, P, Q_1, Q_0)$ *be defined as in Lemma 3.3.8. Assume that*

$$x_1 > \lim_{\lambda_1 \to \infty} \sup_{(P,Q_1,Q_0) \in \mathcal{C}_f} E_{Q_1}\left[\ell\left(L\left(\frac{\lambda(\lambda_1, P, Q_1, Q_0)}{\lambda_1} \frac{dP}{dQ_1}\right)\right)\right],$$

and $\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0))] \geq x_1$.

Then a solution to the maximization problem (3.18) *exists and both constraints are binding. The unique solution is given by*

$$X^* := x^*\left(\lambda_1^* \frac{dQ_1^*}{dQ_0^*}, \lambda_2^* \frac{dP^*}{dQ_0^*}\right),$$

where x^* *is defined as in* (3.20). *Furthermore,* P^*, Q_1^*, *and* Q_0^* *are worst case measures, i.e., they satisfy* (3.69), (3.70), *and* (3.71), *and the utility of the optimal claim is given by*

$$\inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X^*)] = v_{\lambda_1^*, \lambda_2^*}(P^* | Q_1^* | Q_0^*) + \lambda_1^* x_1 + \lambda_2^* x_0. \tag{3.72}$$

This theorem provides a solution to the robust utility maximization problem (3.18) under both a budget and a risk constraint. The solution is of the same form as the one to Problem (3.15) without model uncertainty. Furthermore, there exists a self-financing strategy that super-replicates the optimal contingent claim X^* by Lemma 2.3.4. If the worst case martingale measures $\tilde{P}$, $\hat{P}$, and P^* in the cases (ii), (iii), and (iv), respectively, are indeed equivalent martingale measures, then the optimal claim is even attainable by some self-financing strategy. This follows from Theorem 3.2 by Ansel and Stricker [1994]. The optimal claim can be represented as a portfolio of the optimal claim of a problem without risk constraint and two puts with strike $\bar{x}_\ell$, in the same way as in Theorem 3.2.3.

Note that in case (ii), the robust problem (3.18) has the same solution as the classical problem (3.15) under $\tilde{Q}_1$ and $\tilde{P}$, and these two measures may be interpreted as worst case measures for the utility maximization problem. In case (iii), the robust problem (3.18) can be reduced to a utility maximization problem with utility functional $E_{\hat{Q}_0}[u(X)]$ and budget constraint $E_{\hat{P}}[X]$. The risk constraint is automatically satisfied in this case, and $\hat{P}$ and $\hat{Q}_0$ are worst case measures for the optimal claim. In the last case (iv), X^* is the solution to the utility maximization problem (3.15) with a joint budget and risk constraint under the measures Q_0^*, Q_1^*, and P^*. Hence, in general we have to deal with different subjective and martingale measures in each of the cases (ii) to (iv), and the robust problem cannot completely be reduced to a classical problem.

Remark 3.3.14. *The condition*

$$x_1 > \lim_{\lambda_1 \to \infty} \sup_{(P,Q_1,Q_0)\in\mathcal{C}_f} E_{Q_1}\left[\ell\left(L\left(\frac{\lambda(\lambda_1,P,Q_1,Q_0)}{\lambda_1}\frac{dP}{dQ_1}\right)\right)\right] \tag{3.73}$$

in case (iv) is not completely satisfactory. From an economic point of view one would prefer a condition of the form

$$x_1 > \sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}\left[\ell\left(-Y^*\right)\right)], \tag{3.74}$$

that is, the risk limit x_1 has to be larger than the robust loss of the loss-minimizing claim Y^*. If we could prove in Lemma 3.3.8 that a pair of Lagrange multipliers $(\lambda_1^*, \lambda_2^*)$ exists if $x_0 \geq \bar{x}_\ell$ or, if $x_0 < \bar{x}_\ell$ and Condition (3.74) *holds, then we would indeed have treated all possible cases in Theorem 3.3.13, since the case $x_1 \leq \sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-Y^*))]$ and $x_0 < \bar{x}_\ell$ is covered in (i) and (ii). As things stand so far, we were not able yet to give a solution for the case*

$$\begin{aligned}\sup_{Q_1\in\mathcal{Q}_1} E_{Q_1}[\ell(-Y^*)] &< x_1 \\ &\leq \lim_{\lambda_1 \to \infty} \sup_{(P,Q_1,Q_0)\in\mathcal{C}_f} E_{Q_1}\left[\ell\left(L\left(\frac{\lambda(\lambda_1,P,Q_1,Q_0)}{\lambda_1}\frac{dP}{dQ_1}\right)\right)\right]\end{aligned}$$

if $x_0 < \bar{x}_\ell$, *and for the case*

$$0 < x_1 \leq \lim_{\lambda_1 \to \infty} \sup_{(P,Q_1,Q_0) \in \mathcal{C}_f} E_{Q_1}\left[\ell\left(L\left(\frac{\lambda(\lambda_1, P, Q_1, Q_0)}{\lambda_1} \frac{dP}{dQ_1}\right)\right)\right]$$

if $x_0 \geq \bar{x}_\ell$.

In order to illustrate this gap also from a mathematical point of view, note that for $(P, Q_1, Q_0) \in \mathcal{C}_f$, *Lemma 3.2.17 implies*

$$\lim_{\lambda_1 \to \infty} E_{Q_1}\left[\ell\left(L\left(\frac{\lambda(\lambda_1, P, Q_1, Q_0)}{\lambda_1} \frac{dP}{dQ_1}\right)\right)\right] = E_{Q_1}\left[\ell\left(L\left(c_{P,Q_1} \frac{dP}{dQ_1}\right)\right)\right]$$

if $x_0 < \bar{x}_\ell$ *and*

$$\lim_{\lambda_1 \to \infty} E_{Q_1}\left[\ell\left(L\left(\frac{\lambda(\lambda_1, P, Q_1, Q_0)}{\lambda_1} \frac{dP}{dQ_1}\right)\right)\right] = 0$$

if $x_0 \geq \bar{x}_\ell$.

Furthermore,

$$\begin{aligned}
-\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-Y^*)] &= \tilde{v}_{c^*}(\tilde{P}|\tilde{Q}_1) + c^* x_0 \\
&= \inf_{P \in \mathcal{P}^T} \inf_{Q_1 \in \mathcal{Q}_1} \inf_{c>0} \{\tilde{v}_c(P|Q_1) + c x_0\} \\
&= \inf_{P \in \mathcal{P}^T} \inf_{Q_1 \in \mathcal{Q}_1} \left\{\tilde{v}_{c_{P,Q_1}}(P|Q_1) + c_{P,Q_1} x_0\right\} \\
&= -\sup_{P \in \mathcal{P}^T} \sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}\left[\ell\left(L\left(c_{P,Q_1} \frac{dP}{dQ_1}\right)\right)\right].
\end{aligned}$$

The first equality follows from Proposition 3.3.6, the second and the third from our choice of c^*, c_{P,Q_1}, $\tilde{P}$, *and* $\tilde{Q}_1$, *and the last one form Corollary 3.2.11.*

Hence we should try to show that we are allowed to interchange the limit and the supremum in (3.73) *in order to treat all possible cases in Theorem 3.3.13.*

Proof of Theorem 3.3.13. (i) follows from Proposition 3.3.6.

(ii) Here the reasoning is the same as in the proof of Theorem 3.2.3(ii): X^* solves the loss minimization problem (3.66) by Proposition 3.3.6. Hence it satisfies both constraints, and by Proposition 3.3.6, any other contingent claim satisfying both constraints equals X^* on the set $\{d\tilde{P}/dR > 0\}$. On $\{d\tilde{P}/dR = 0\}$ we cannot do any better than setting X^* equal to ∞. Hence, X^* solves the utility maximization problem (3.18), and it is the unique solution on the set $\{d\tilde{P}/dR > 0\}$.

In order to show (iii) and (iv), take a contingent claim $X \in \mathcal{X}(x_0, x_1)$ that satisfies the constraints, and $\lambda_1 \geq 0$, $\lambda_2 > 0$. Then (3.32) in the proof of Theorem 3.2.3 applied to $P' \in \mathcal{P}^T$, $Q'_1 \in \mathcal{Q}_1$, and $Q'_0 \in \mathcal{Q}_0$ with $v_{1,1}(P'|Q'_1|Q'_0) < \infty$ implies

$$\begin{aligned}
\inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X)] &\leq E_{Q'_0}[u(X)] \\
&\leq v_{\lambda_1,\lambda_2}(P'|Q'_1|Q'_0) + \lambda_1 x_1 + \lambda_2 x_0 \\
&= E_{Q'_0}\left[u\left(x'\left(\lambda_1 \frac{dQ'_1}{dQ'_0}, \lambda_2 \frac{dP'}{dQ'_0}\right)\right)\right] \\
&\quad + \quad \lambda_1 \left(x_1 - E_{Q'_1}\left[\ell\left(-x'\left(\lambda_1 \frac{dQ'_1}{dQ'_0}, \lambda_2 \frac{dP'}{dQ'_0}\right)\right)\right]\right) \\
&\quad + \quad \lambda_2 \left(x_0 - E_{P'}\left[x'\left(\lambda_1 \frac{dQ'_1}{dQ'_0}, \lambda_2 \frac{dP'}{dQ'_0}\right)\right]\right).
\end{aligned} \tag{3.75}$$

(iii) Let $P' = \hat{P}$, $Q'_0 = \hat{Q}_0$ in (3.75). If $\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0))] < x_1$, then the last two summands in (3.75) are equal to zero for $\lambda_1 = 0$, $\lambda_2 = \tilde{\lambda}_2$. Since $x^*(0, y_2) = I(y_2)$, this implies

$$\sup_{X \in \mathcal{X}(x_0,x_1)} \inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X)] \leq E_{\hat{Q}_0}\left[u\left(I\left(\tilde{\lambda}_2 \frac{d\hat{P}}{d\hat{Q}_0}\right)\right)\right].$$

By Proposition 2.3.8 in the previous chapter the right-hand side is equal to $\inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0))]$, and $I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0)$ satisfies the budget constraint. Thus, $I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0)$ is a solution to Problem (3.18), and the UBSR constraint is not binding.

The uniqueness follows in the same way as in the proof of Theorem 2.3.9 and from the fact that $\hat{Q}_0 \sim R$ by assumption: Assume that $\tilde{X} \in \mathcal{X}(x_0, x_1)$ solves Problem (3.18). Then we have $\sup_{P \in \mathcal{P}^T} E_P[\tilde{X}] \leq x_0$ and $\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-\tilde{X})] \leq x_1$, and hence

$$\inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(\tilde{X})] \leq E_{\hat{Q}_0}[u(\tilde{X})] \leq E_{\hat{Q}_0}[u(X^*)].$$

The second inequality holds strictly unless $\tilde{X} = X^*$ $\hat{Q}_0$- and hence R-almost surely. This follows from the fact that in this case X^* is the solution to Problem (3.15) under $\hat{P}$ and $\hat{Q}_0$ and from the uniqueness result in Theorem 3.2.3. But the strict inequality is a contradiction to $E_{\hat{Q}_0}[u(X^*)] = \inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X^*)]$. Thus $\tilde{X} = X^*$ R-almost surely.

(iv) Let $P' = P^*$, $Q_1' = Q_1^*$, and $Q_0' = Q_0^*$. Since $(\lambda_1^*, \lambda_2^*)$ minimizes $v_{\lambda_1,\lambda_2}(P^*|Q_1^*|Q_0^*) + \lambda_1 x_1 + \lambda_2 x_0$ it follows from Corrolary 3.2.11 that the two terms in the brackets on the right-hand side of (3.75) equal zero for $\lambda_1 = \lambda_1^*$ and $\lambda_2 = \lambda_2^*$. Proposition 3.3.12 implies that X^* satisfies the constraints and that $E_{Q_0^*}[u(X^*)] = \inf_{Q_0 \in \mathcal{Q}_0} E_{Q_0}[u(X^*)]$. This concludes the proof of (3.72) and of the optimality of X^*. Both constraints are binding due to the assumption $\sup_{Q_1 \in \mathcal{Q}_1} E_{Q_1}[\ell(-I(\tilde{\lambda}_2 d\hat{P}/d\hat{Q}_0))] \geq x_1$. Furthermore, in this case, the robust utility maximization problem is equivalent to the classical problem with $\mathcal{Q}_0 = \{Q_0^*\}$. Now the uniqueness follows in the same way as in (iii). □

3.3.1 Proofs

This section contains the proofs of the Lemmata 3.3.3 and 3.3.8, and of the Propositions 3.3.9 and 3.3.12.

Proof of Lemma 3.3.3. Note that v and $\tilde{v}$ as suprema of linear functions are convex. Let us start with proving that $G(\lambda_1)$ is convex. Let $\lambda_1^i \geq 0$ $(i = 1, 2)$ and let $\epsilon > 0$ be fixed. Choose $\lambda_2^i > 0$, $Q_1^i \in \mathcal{Q}_1$, $Q_0^i \in \mathcal{Q}_0$, and $P^i \in \mathcal{P}^T$ such that

$$G(\lambda_1^i) \geq v_{\lambda_1^i,\lambda_2^i}(P^i|Q_1^i|Q_0^i) + \lambda_2^i x_0 + \lambda_1^i x_1 - \epsilon.$$

Let $\alpha \in (0, 1)$. Since the sets $\mathcal{P}^T$, $\mathcal{Q}_1$, and $\mathcal{Q}_0$ are convex,

$$\tilde{P} := \frac{\alpha\lambda_2^1 P^1 + (1-\alpha)\lambda_2^2 P^2}{\alpha\lambda_2^1 + (1-\alpha)\lambda_2^2} \in \mathcal{P}^T, \qquad \tilde{Q}_1 := \frac{\alpha\lambda_1^1 Q_1^1 + (1-\alpha)\lambda_1^2 Q_1^2}{\alpha\lambda_1^1 + (1-\alpha)\lambda_1^2} \in \mathcal{Q}_1,$$

and $\alpha Q_0^1 + (1-\alpha) Q_0^2 \in \mathcal{Q}_0$. Let $\phi := dP/dR$, $\tilde{\phi} := d\tilde{P}/dR$, $\psi_j^i := dQ_j^i/dR$ $(i = 1, 2,\ j = 0, 1)$, and $\tilde{\psi}_1 := d\tilde{Q}_1/dR$. Then

$$\begin{aligned}
&G(\alpha\lambda_1^1 + (1-\alpha)\lambda_1^2)\\
&\leq E_R\left[v\left([\alpha\lambda_2^1 + (1-\alpha)\lambda_2^2]\cdot\tilde{\phi},\ [\alpha\lambda_1^1 + (1-\alpha)\lambda_1^2]\cdot\tilde{\psi}_1,\ \alpha\psi_0^1 + (1-\alpha)\psi_0^2\right)\right]\\
&\qquad + [\alpha\lambda_2^1 + (1-\alpha)\lambda_2^2]\cdot x_0 + [\alpha\lambda_1^1 + (1-\alpha)\lambda_1^2]\cdot x_1\\
&= E_R\left[v\left(\alpha\lambda_2^1\phi^1 + (1-\alpha)\lambda_2^2\phi^2,\ \alpha\lambda_1^1\psi_1^1 + (1-\alpha)\lambda_1^2\psi_1^2,\ \alpha\psi_0^1 + (1-\alpha)\psi_0^2\right)\right]\\
&\qquad + [\alpha\lambda_2^1 + (1-\alpha)\lambda_2^2]\cdot x_0 + [\alpha\lambda_1^1 + (1-\alpha)\lambda_1^2]\cdot x_1\\
&\leq E_R\left[\alpha v\left(\lambda_2^1\phi^1,\ \lambda_1^1\psi_1^1,\ \psi_0^1\right) + (1-\alpha)v\left(\lambda_2^2\phi^2,\ \lambda_1^2\psi_1^2,\ \psi_0^2\right)\right]\\
&\qquad + [\alpha\lambda_2^1 + (1-\alpha)\lambda_2^2]\cdot x_0 + [\alpha\lambda_1^1 + (1-\alpha)\lambda_1^2]\cdot x_1\\
&= \alpha\left(v_{\lambda_1^1,\lambda_2^1}(P^1|Q_1^1|Q_0^1) + \lambda_2^1 x_0 + \lambda_1^1 x_1\right)\\
&\qquad + (1-\alpha)\left(v_{\lambda_1^2,\lambda_2^2}(P^2|Q_1^2|Q_0^2) + \lambda_2^2 x_0 + \lambda_1^2 x_1\right)\\
&\leq \alpha\, G(\lambda_1^1) + (1-\alpha)G(\lambda_1^2) + \epsilon.
\end{aligned}$$

Since ϵ was arbitrary, the proof of the convexity of G is complete.

The convexity of H, $\tilde{H}$, and G_{P,Q_1,Q_0} can be shown in the same way. $\square$

Proof of Lemma 3.3.8. Note that the limit

$$\lim_{\lambda\to\infty} \sup_{(P,Q_1,Q_0)\in\mathcal{C}_f} E_{Q_1}\left[\ell\left(L\left(\frac{\lambda(\lambda_1,P,Q_1,Q_0)}{\lambda_1}\frac{dP}{dQ_1}\right)\right)\right]$$

exists because $\lambda(\lambda_1,P,Q_1,Q_0)/\lambda_1$ is decreasing in λ_1 by Lemma 3.2.14, and ℓ and L are increasing.

We split this proof into two steps. In the first step we show that for any $\lambda_1 \geq 0$, there is a finite minimizer $\lambda_2 = \lambda(\lambda_1) > 0$ of the convex function

$$\lambda_2 \mapsto \inf_{P\in\mathcal{P}^T} \inf_{Q_1\in\mathcal{Q}_1} \inf_{Q_0\in\mathcal{Q}_0} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0.$$

In the second step we then show that under the assumptions of the lemma, the convex function

$$\lambda_1 \mapsto \inf_{P\in\mathcal{P}^T} \inf_{Q_1\in\mathcal{Q}_1} \inf_{Q_0\in\mathcal{Q}_0} \inf_{\lambda_2>0} \{v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0\}$$

converges to infinity as λ_1 tends to infinity. Thus, it assumes its infimum in some finite value $\lambda_1^* \geq 0$. This implies that

$$\inf_{P\in\mathcal{P}^T} \inf_{Q_1\in\mathcal{Q}_1} \inf_{Q_0\in\mathcal{Q}_0} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0$$

achieves its minimum in $(\lambda_1^*, \lambda_2^*)$ with $\lambda_2^* := \lambda(\lambda_1^*)$.

Step 1. Let $\lambda_1 \geq 0$ be fixed. Since

$$\lambda_2 \mapsto \inf_{P\in\mathcal{P}^T} \inf_{Q_1\in\mathcal{Q}_1} \inf_{Q_0\in\mathcal{Q}_0} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0$$

is convex in λ_2 by Lemma 3.3.3, it remains to show that its minimizer is strictly positive and finite. It can be shown in the same way as in Lemma 2.1.6 that $\lim_{\lambda_2\to\infty} v(\lambda_2,\lambda_1,1)/\lambda_2 = \bar{x}_u = 0$. Hence Jensen's inequality implies

$$\begin{aligned}
&\lim_{\lambda_2\to\infty}\left(\inf_{P\in\mathcal{P}^T} \inf_{Q_1\in\mathcal{Q}_1} \inf_{Q_0\in\mathcal{Q}_0} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0\right)\\
&\quad\geq \lim_{\lambda_2\to\infty}\left(v(\lambda_2,\lambda_1,1) + \lambda_1 x_1 + \lambda_2 x_0\right)\\
&\quad= \lim_{\lambda_2\to\infty} \lambda_2\left(\frac{v(\lambda_2,\lambda_1,1)}{\lambda_2} + x_0\right) + \lambda_1 x_1 = \infty,
\end{aligned}$$

due to $x_0 > 0$. By the same inequality as above we have

$$\lim_{\lambda_2 \searrow 0} \left(\inf_{P \in \mathcal{P}^T} \inf_{Q_1 \in \mathcal{Q}_1} \inf_{Q_0 \in \mathcal{Q}_0} v_{\lambda_1, \lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0 \right) \\ \geq u(\infty) - \lambda_1 \ell(-\infty) + \lambda_1 x_1.$$

If $\lim_{\lambda_2 \to 0} v(\lambda_2, \lambda_1, 1) = u(\infty) - \lambda_1 \ell(-\infty) = \infty$, then it is obvious that the minimizing value λ_2 is positive. If $\lim_{\lambda_2 \to 0} v(\lambda_2, \lambda_1, 1) < \infty$, assume that the infimum is achieved in $\lambda_2 = 0$. Choose $P \in \mathcal{P}^T$, $Q_1 \in \mathcal{Q}_1$, and $Q_0 \in \mathcal{Q}_0$ such that $v_{1,1}(P|Q_1|Q_0) < \infty$ and hence $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) < \infty$ for all $\lambda_1 \geq 0$ and $\lambda_2 > 0$ by Remark 3.2.8(ii). This implies

$$u(\infty) - \lambda_1 \ell(-\infty) + \lambda_1 x_1 \leq \inf_{P \in \mathcal{P}^T} \inf_{Q_1 \in \mathcal{Q}_1} \inf_{Q_0 \in \mathcal{Q}_0} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0$$

$$\leq v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0$$

$$\leq v_{\lambda_1,\epsilon}(P|Q_1|Q_0) + \lambda_1 x_1 + (\lambda_2 - \epsilon) \left(x_0 - E_P \left[x^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda_2 \frac{dP}{dQ_0} \right) \right] \right)$$

for all $\lambda_2 > \epsilon > 0$. The last inequality follows from the convexity of $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0)$ in $\lambda_2 > 0$ and Lemma 3.2.10. $v_{\lambda_1,\epsilon}(P|Q_1|Q_0)$ increases to $u(\infty) - \lambda_1 \ell(-\infty)$ as ϵ converges to zero since $v(\epsilon dP/dR, \lambda_1 dQ_1/dR, dQ_0/dR) \in L^1(R)$ for all $\epsilon > 0$. But this leads to a contradiction since the expectation on the right-hand side converges to infinity as $\lambda_2 \to 0$ due to the Inada condition (2.7), Lemma 3.2.9(i), and Lemma 3.2.12(vi). Hence the convex function $\inf_{P \in \mathcal{P}^T} \inf_{Q_1 \in \mathcal{Q}_1} \inf_{Q_0 \in \mathcal{Q}_0} v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0$ achieves its minimum in some finite value $\lambda_2^* > 0$.

Step 2. It remains to show that under the conditions of the lemma the convex function

$$G(\lambda_1) = \inf_{P \in \mathcal{P}^T} \inf_{Q_1 \in \mathcal{Q}_1} \inf_{Q_0 \in \mathcal{Q}_0} \inf_{\lambda_2 > 0} v_{\lambda_1,\lambda_2} \left\{ (P|Q_1|Q_0) + \lambda_1 x_1 + \lambda_2 x_0 \right\}$$

converges to infinity as $\lambda_1 \to \infty$ and hence $\lambda_1^* < \infty$. For $\lambda_1 \geq 0$ and $(P, Q_1, Q_0) \in \mathcal{C}_f$, define $\lambda(\lambda_1, P, Q_1, Q_0)$ as the minimizing value of the convex function $\lambda_2 \mapsto v_{\lambda_1,\lambda_2}(P|Q_1|Q_0) + \lambda_2 x_0$. By Lemma 3.2.10 we have

$$x_0 = E_P \left[x^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda(\lambda_1, P, Q_1, Q_0) \frac{dP}{dQ_0} \right) \right].$$

Let

$$G_{P,Q_1,Q_0}(\lambda_1) = v_{\lambda_1, \lambda(\lambda_1, P, Q_1, Q_0)}(P|Q_1|Q_0) + \lambda_1 x_1 + \lambda(\lambda_1, P, Q_1, Q_0) \cdot x_0$$

for $(P, Q_1, Q_0) \in \mathcal{C}_f$. Then by Lemma 3.2.10 and the convexity of the function $v_{\lambda_1,\lambda_2}(P|Q_1|Q_0)$ in λ_1 and λ_2,

$$\begin{aligned}
&G_{P,Q_1,Q_0}(\lambda_1 + h) - G_{P,Q_1,Q_0}(\lambda_1) \\
&= v_{\lambda_1+h,\lambda(\lambda_1+h,P,Q_1,Q_0)}(P|Q_1|Q_0) - v_{\lambda_1,\lambda(\lambda_1,P,Q_1,Q_0)}(P|Q_1|Q_0) \\
&\qquad + h \cdot x_1 + (\lambda(\lambda_1 + h, P, Q_1, Q_0) - \lambda(\lambda_1, P, Q_1, Q_0)) \cdot x_0 \\
&\geq (\lambda(\lambda_1 + h, P, Q_1, Q_0) - \lambda(\lambda_1, P, Q_1, Q_0)) \\
&\qquad \cdot \left(x_0 - E_P \left[x^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda(\lambda_1, P, Q_1, Q_0) \frac{dP}{dQ_0} \right) \right] \right) \\
&\qquad + h \left(x_1 - E_{Q_1} \left[\ell \left(-x^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda(\lambda_1, P, Q_1, Q_0) \frac{dP}{dQ_0} \right) \right) \right] \right) \\
&= h \left(x_1 - E_{Q_1} \left[\ell \left(-x^* \left(\lambda_1 \frac{dQ_1}{dQ_0}, \lambda(\lambda_1, P, Q_1, Q_0) \frac{dP}{dQ_0} \right) \right) \right] \right) \\
&\geq h \left(x_1 - E_{Q_1} \left[\ell \left(L \left(\frac{\lambda(\lambda_1, P, Q_1, Q_0)}{\lambda_1} \frac{dP}{dQ_1} \right) \right) \right] \right),
\end{aligned}$$

where the last inequality follows from Lemma 3.2.12(ix).

This implies

$$\begin{aligned}
&G(\lambda_1 + h) - G(\lambda_1) \\
&\qquad \geq h \left(x_1 - \sup_{(P,Q_1,Q_0) \in \mathcal{C}_f} E_{Q_1} \left[\ell \left(L \left(\frac{\lambda(\lambda_1, P, Q_1, Q_0)}{\lambda_1} \frac{dP}{dQ_1} \right) \right) \right] \right).
\end{aligned}$$

But the term on the right-hand side is strictly positive by assumption if λ_1 is large enough, hence G is finally increasing and assumes its infimum in some finite value λ_1^*. □

For the proof of Proposition 3.3.9, we need the following auxiliary result. In order to simplify the notations, we define $f(\phi, \psi_1, \psi_0) := v(\lambda_2^* p, \lambda_1^* \psi_1, \psi_0)$ and $f(P|Q_1|Q_0) := v_{\lambda_1^*,\lambda_2^*}(P|Q_1|Q_0)$.

Lemma 3.3.15. *The set*

$$\left\{ f \left(\left(\frac{dP}{dR} + \epsilon \right), \frac{dQ_1}{dR}, \frac{dQ_0}{dR} \right)^- : P \in \mathcal{P}^T, Q_1 \in \mathcal{Q}_1, Q_0 \in \mathcal{Q}_0 \right\}$$

is uniformly integrable with respect to R.

Proof. We obtain from the proof of Theorem 1.2.8 that

$$\left\{ \sup_{x>0} \left\{ \frac{dQ_0}{dR} u(x) - \lambda_2 x \left(\frac{dP}{dR} + \epsilon \right) \right\}^- : P \in \mathcal{P}^T, Q_0 \in \mathcal{Q}_0 \right\}$$

is uniformly integrable. Now the result follows from

$$\begin{aligned} f&\left(\left(\frac{dP}{dR} + \epsilon \right), \frac{dQ_1}{dR}, \frac{dQ_0}{dR} \right) \\ &= \sup_{x>0} \left\{ \frac{dQ_0}{dR} u(x) - \lambda_1 \ell(-x) \frac{dQ_1}{dR} - \lambda_2 x \left(\frac{dP}{dR} + \epsilon \right) \right\} \\ &\geq \sup_{x>0} \left\{ \frac{dQ_0}{dR} u(x) - \lambda_2 x \left(\frac{dP}{dR} + \epsilon \right) \right\} - \lambda_1 \ell(0) \frac{dQ_1}{dR}, \end{aligned}$$

the uniform integrability of $\mathcal{K}_{\mathcal{Q}_1}$ due to Assumption 3.1.3, and the fact that the sum of two uniformly integrable sets is again uniformly integrable. □

Proof of Proposition 3.3.9. We may assume without loss of generality that we have
$\inf_{P \in \mathcal{P}} \inf_{Q_1 \in \mathcal{Q}_1} \inf_{Q_0 \in \mathcal{Q}_0} f(P|Q_1|Q_0) < \infty$ since otherwise, any $(P, Q_1, Q_0) \in \mathcal{P} \times \mathcal{Q}_1 \times \mathcal{Q}_0$ is a minimizer of the generalized divergence. Note that the function $f(\phi, \psi_1, \psi_0)$ is continuous on $[0, \infty) \times [0, \infty) \times (0, \infty)$ since the functions g and x^* from Lemma 3.2.12 are continuous and $f(\phi, \psi_1, \psi_0) = \psi_0 g(x^*(\lambda_1 \psi_1 / \psi_0, \lambda_2 \phi / \psi_0))$. Now the proof follows the lines of the one of Theorem 1.2.8: Let $(Q_0^n)_{n \geq 1} \subseteq \mathcal{Q}_0$, $(Q_1^n)_{n \geq 1} \subseteq \mathcal{Q}_1$, and $(P_n)_{n \geq 1} \subseteq \mathcal{P}^T$ be such that $f(P^n|Q_1^n|Q_0^n)$ converges to the infimum of the values $f(P|Q_1|Q_0)$ over $P \in \mathcal{P}^T$, $Q_1 \in \mathcal{Q}_1$ and $Q_0 \in \mathcal{Q}_0$, and define

$$\psi_i^n := \frac{dQ_i^n}{dR}$$

for $i = 0, 1$. By Delbaen and Schachermayer [1994], Lemma A1.1, we can choose

$$\psi_i^{n,0} \in \operatorname{conv}(\psi_i^n, \psi_i^{n+1}, \ldots) \quad (n = 1, 2, \ldots)$$

and functions ψ_i^* such that

$$\psi_i^{n,0} \longrightarrow \psi_i^* \qquad R-\text{almost surely.}$$

Since the sets $\mathcal{K}_{\mathcal{Q}_i}$ are weakly compact we have $\psi_i^* \in \mathcal{K}_{\mathcal{Q}_i}$, i.e., ψ_i^* are the densities of some measures $Q_i^* \in \mathcal{Q}_i$. Due to Lemma 1.2.7, we can also choose

$$P^{n,0} \in \operatorname{conv}(P^n, P^{n+1}, \ldots) \quad (n = 1, 2, \ldots)$$

and $P^* \in \mathcal{P}^T$ such that

$$\frac{dP^{n,0}}{dR} \longrightarrow \frac{dP^*}{dR} \qquad R-\text{almost surely.} \tag{3.76}$$

Define $\phi^{n,0} := dP^{n,0}/dR$ and $\phi^* := dP^*/dR$. Note first that

$$\begin{aligned} f(P^*|Q_1^*|Q_0^*) &= E_R\left[f\left(\phi^*, \psi_1^*, \psi_0^*\right)\right] \\ &= E_R\left[\lim_{\epsilon\to 0} f\left(\phi^* + \epsilon, \psi_1^*, \psi_0^*\right)\right] \\ &= \lim_{\epsilon\to 0} E_R\left[f\left(\phi^* + \epsilon, \psi_1^*, \psi_0^*\right)\right] \end{aligned}$$

by monotone convergence, since $f(\cdot, \psi_1, \psi_0)$ is continuous and decreasing on $[0, \infty)$, and

$$E_R\left[f\left(\phi^* + \epsilon, \psi_1^*, \psi_0^*\right)\right] \geq f(E_R[\phi^*] + \epsilon, 1, 1) > -\infty$$

by definition of f as a supremum. Lemma 3.3.15 implies

$$\begin{aligned} E_R\left[f\left(\phi^* + \epsilon, \psi_1^*, \psi_0^*\right)\right] &= E_R\left[\lim_{n\to\infty} f(\phi^{n,0} + \epsilon, \psi_1^{n,0}, \psi_0^{n,0})\right] \\ &= E_R\left[\lim_{n\to\infty} f^+(\phi^{n,0} + \epsilon, \psi_1^{n,0}, \psi_0^{n,0})\right] \\ &\quad - E_R\left[\lim_{n\to\infty} f^-(\phi^{n,0} + \epsilon, \psi_1^{n,0}, \psi_0^{n,0})\right] \\ &\leq \liminf_{n\to\infty} E_R[f(\phi^{n,0} + \epsilon, \psi_1^{n,0}, \psi_0^{n,0})] \\ &\leq \liminf_{n\to\infty} E_R[f(\phi^{n,0}, \psi_1^{n,0}, \psi_0^{n,0})] \\ &\leq \liminf_{n\to\infty} E_R[f(\phi^{n}, \psi_1^{n}, \psi_0^{n})] = \inf_{P\in\mathcal{P}^T} \inf_{Q_1\in\mathcal{Q}_1} \inf_{Q_0\in\mathcal{Q}_0} f(P|Q_1|Q_0). \end{aligned}$$

The first equality follows from the continuity of $f(\cdot+\epsilon, \cdot, \cdot)$ on $[0, \infty)^2 \times (0, \infty)$, the first inequality follows from Fatou's lemma (applied to the first term) and Lebesgue's theorem (applied to the second term) due to Lemma 3.3.15, and the last one from the convexity of $f(\cdot, \cdot, \cdot)$. This shows that $f(\cdot| \cdot |\cdot)$ attains its minimum in (P^*, Q_1^*, Q_0^*). $\square$

Proof of Proposition 3.3.12. This can be shown in exactly the same way as Proposition 2.3.8 by defining
(i) $f(\phi) := v(\lambda_2^*\phi, \lambda_1^* dQ_1^*/dR, dQ_0^*/dR)$ for $P \in \mathcal{P}^T$ and $\phi := dP/dR$,
(ii) $f(\psi_1) := v(\lambda_2^* dP^*/dR, \lambda_1^*\psi_1, dQ_0^*/dR)$ for $Q_1 \in \mathcal{Q}_1$ and $\psi_1 := dQ_1/dR$,
(iii) $f(\psi_0) := v(\lambda_2^* dP^*/dR, \lambda_1^* dQ_1^*/dR, \psi_0)$ for $Q_0 \in \mathcal{Q}_0$ and $\psi_0 := dQ_0/dR$,
and using Lemma 3.2.12(x).

Note that in (i) for any $P \in \mathcal{P}^T$ there is $\alpha \in (0,1]$ such that $v_{\lambda_1^*,\lambda_2^*}(\alpha P + (1-\alpha)P^*|Q_1^*|Q_0^*) < \infty$. Indeed, let $P \in \mathcal{P}^T$, $\alpha \in (0,1)$, and define $\phi^* := dP^*/dR$, $\phi := dP/dR$, $\phi^\alpha := \alpha\phi + (1-\alpha)\phi^*$, $\psi_1^* := dQ_1^*/dR$, and $\psi_0^* := dQ_0^*/dR$. The convex function $f(\phi) := v(\lambda_2^*\phi, \lambda_1^*\psi_1^*, \psi_0^*)$ has increasing derivative $f'(\phi) = -\lambda_2^* x^*(\lambda_1^*\psi_1^*/\psi_0^*, \lambda_2^*\phi/\psi_0^*) \le 0$ on $\{\phi > 0\}$. Hence we obtain on $\{\phi^\alpha > 0\}$,

$$\begin{aligned} f(\phi^\alpha) &\le f(\phi^*) - f'(\phi^\alpha)(\phi^* - \phi^\alpha) \\ &\le f(\phi^*) - \lambda_2^* f'((1-\alpha)\phi^*)\phi^* \\ &= f(\phi^*) + \lambda_2^* x^* \left(\lambda_1^* \frac{\psi_1^*}{\psi_0^*}, (1-\alpha)\lambda_2^* \frac{\phi^*}{\psi_0^*}\right)\phi^*, \end{aligned}$$

which is in $L^1(R)$ due to Assumption 3.3.1 and Lemma 3.2.9(i). If $f(0) = u(\infty) - \lambda_1^*\ell(-\infty) = \infty$, then $R(\phi^\alpha > 0) = 1$ since $E_R[f(\phi^*)] < \infty$. Otherwise $v_{\lambda_1^*,\lambda_2^*}(\alpha P + (1-\alpha)P^*|Q_1^*|Q_0^*) = E_R[f(\phi^\alpha); \phi^\alpha > 0] + (u(\infty) - \lambda_1^*\ell(-\infty)) \cdot R(\phi^\alpha = 0)$, and the second term is bounded for any $P \in \mathcal{P}^T$.

In the same way, for the proof of (ii) we do not need any assumption on the set $\mathcal{Q}_1$ as Assumption 2.3.2 in Proposition 2.3.8 since for any $Q_1 \in \mathcal{Q}_1$ and $\alpha \in (0,1)$ we have $v_{\lambda_1^*,\lambda_2^*}(P^*|\alpha Q_1^* + (1-\alpha)Q_1|Q_0^*) < \infty$. Indeed, let $Q_1 \in \mathcal{Q}_1$ and define in addition to above $\psi_1 := dQ_1/dR$ and $\psi_1^\alpha := \alpha\psi_1 + (1-\alpha)\psi_1^*$. For the convex function $f(\psi_1) := v(\lambda_2^*\phi^*, \lambda_1^*\psi_1, \psi_0^*)$ with increasing derivative $f'(\psi_1) = -\lambda_1^*\ell(-x^*(\lambda_1^*\psi_1/\psi_0^*, \lambda_2^*\phi^*/\psi_0^*)) \le 0$, we obtain

$$\begin{aligned} f(\psi_1^\alpha) &\le f(\psi_1^*) - f'(\psi_1^\alpha)(\psi_1^* - \psi_1^\alpha) \\ &\le f(\psi_1^*) + \lambda_1^*\ell\left(-x^*\left((1-\alpha)\lambda_1^*\frac{\psi_1^*}{\psi_0^*}, \lambda_2^*\frac{\phi^*}{\psi_0^*}\right)\right)\psi_1^*, \end{aligned}$$

which is in $L^1(R)$ due to Assumption 3.3.1 and Lemma 3.2.9(i). □

3.4 Examples

In the current section we focus on two examples of a financial market with a single risky stock and a bond. We assume that the bond price is constant. The stock price is modelled either as a geometric Brownian motion or a geometric Poisson process. For an exponential utility function, we compare the optimal contingent claim in the UBSR-constrained problem with a binding risk constraint to two benchmark cases: the solution to the classical problem without risk constraint, and the solution to the utility maximization problem if the risk constraint is defined in terms of Value at Risk.

As utility function we choose $u(x) = 1 - e^{-x}$. The loss function shall be given by $\ell(x) = (e^x - e^{-\bar{x}_\ell}) \vee 0$. The deterministic function x^* can then easily be calculated as

$$x^*(y_1, y_2) = \begin{cases} -\log(y_2) + \log(1 + y_1) & \text{if } y_2 > e^{-\bar{x}_\ell} + y_1 e^{-\bar{x}_\ell}, \\ \bar{x}_\ell & \text{if } e^{-\bar{x}_\ell} \le y_2 \le e^{-\bar{x}_\ell} + y_1 e^{-\bar{x}_\ell}, \\ -\log(y_2) & \text{if } y_2 < e^{-\bar{x}_\ell}. \end{cases}$$

3.4.1 A Geometric Brownian Motion Model

In our first example we assume that the stock price $(S_t)_{0 \le t \le T}$ can be described by a generalized geometric Brownian motion under the subjective measure Q_0. To be precise, we assume that $B^0 = (B^0_t)_{0 \le t \le T}$ is a Brownian motion under the measure Q_0. The information filtration shall be generated by B^0. The dynamics of S is described by the stochastic differential equation

$$dS_t = S_t(\sigma_t dB^0_t + \mu^0_t dt) \quad (0 \le t \le T),$$

where the stochastic mean $\mu^0 = (\mu^0_t)_{0 \le t \le T}$ and the volatility $\sigma = (\sigma_t)_{0 \le t \le T}$ with $\sigma_t > 0$ are suitable stochastic processes.

In this case, the financial market is complete, and the density of the unique absolutely continuous and equivalent martingale measure P is given by the stochastic exponential

$$\frac{dP}{dQ_0} = \mathcal{E}\left(-\int_0^T \alpha^0_s B^0_s\right) = \exp\left(-\int_0^T \alpha^0_s B^0_s - \frac{1}{2}\int_0^T (\alpha^0_s)^2 ds\right),$$

where $\alpha^0 := \mu^0/\sigma$.

Let us now define the subjective measure Q_1, which is used for the risk constraint of the utility maximization problem. Let $\mu_1 = (\mu^1_t)_{0 \le t \le T}$ be a suitable stochastic process. Setting $\alpha^1 := \mu^1/\sigma$ and

$$B^1_t := B^0_t + \int_0^t (\alpha^0_s - \alpha^1_s) ds \qquad (0 \le t \le T),$$

we assume that B^1 is a Brownian motion under the measure Q_1. By Girsanov's theorem, this holds true if and only if the Radon-Nikodym density of Q_1 with respect to Q_0 is given by the stochastic exponential

$$\begin{aligned} \frac{dQ_1}{dQ_0} &= \mathcal{E}\left(\int_0^T (\alpha^1_s - \alpha^0_s) dB^0_s\right) \\ &= \exp\left(\int_0^T (\alpha^1_s - \alpha^0_s) dB^0_s - \frac{1}{2}\int_0^T (\alpha^1_s - \alpha^0_s)^2 ds\right). \end{aligned}$$

In terms of the Q_1-Brownian motion B^1, the stock price S can be rewritten as

$$dS_t = S_t(\sigma_t dB_t^1 + \mu_t^1 dt) \quad (0 \le t \le T).$$

Define

$$\begin{aligned} A &:= \left\{ \lambda_2^* \frac{dP}{dQ_0} > e^{-\bar{x}_\ell} + \lambda_1^* \frac{dQ_1}{dQ_0} e^{-\bar{x}_\ell} \right\}, \\ B &:= \left\{ e^{-\bar{x}_\ell} \le \lambda_2^* \frac{dP}{dQ_0} \le e^{-\bar{x}_\ell} + \lambda_1^* \frac{dQ_1}{dQ_0} e^{-\bar{x}_\ell} \right\}, \\ C &:= \left\{ \lambda_2^* \frac{dP}{dQ_0} < e^{-\bar{x}_\ell} \right\}, \end{aligned}$$

where λ_1^* and λ_2^* have to be chosen such that X^* below satisfies the budget constraint. By Theorem 3.2.3 we obtain as optimal contingent claim

$$\begin{aligned} X^* &= x^* \left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0} \right) \\ &= \begin{cases} -\log\left(\lambda_2^* \frac{dP}{dQ_0}\right) + \log\left(1 + \lambda_1^* \frac{dQ_1}{dQ_0}\right) & \text{on } A, \\ \bar{x}_\ell & \text{on } B, \\ -\log\left(\lambda_2^* \frac{dP}{dQ_0}\right) & \text{on } C, \end{cases} \\ &= \Bigg[-\log(\lambda_2^*) + \int_0^T \alpha_s^0 dB_s^0 + \frac{1}{2} \int_0^T (\alpha_s^0)^2 ds \\ &\quad + \log\left(1 + \lambda_1^* \exp\left(\int_0^T (\alpha_s^1 - \alpha_s^0) dB_s^0 - \frac{1}{2} \int_0^T (\alpha_s^1 - \alpha_s^0)^2 ds \right)\right) \Bigg] \cdot 1_A \\ &\quad + \bar{x}_\ell \cdot 1_B \\ &\quad + \left[-\log(\lambda_2^*) + \int_0^T \alpha_s^0 dB_s^0 + \frac{1}{2} \int_0^T (\alpha_s^0)^2 ds \right] \cdot 1_C. \end{aligned}$$

Under the VaR constraint $Q_1(X < \bar{x}_\ell) < \alpha$ the solution X^{VaR} can be found in Basak and Shapiro [2001]:

$$\begin{aligned} X^{VaR} &= \begin{cases} \bar{x}_\ell & \text{on } \left\{ e^{-\bar{x}_\ell} \le \lambda^* \frac{dP}{dQ_0} \le \lambda^* \bar{x} \right\}, \\ -\log\left(\lambda^* \frac{dP}{dQ_0}\right) & \text{on the remaining space,} \end{cases} \\ &= \begin{cases} \bar{x}_\ell & \text{on } \left\{ e^{-\bar{x}_\ell} \le \lambda^* \frac{dP}{dQ_0} \le \lambda^* \bar{x} \right\}, \\ -\log(\lambda^*) + \int_0^T \alpha_s^0 dB_s^0 + \frac{1}{2} \int_0^T (\alpha_s^0)^2 ds & \text{on the remaining space,} \end{cases} \end{aligned}$$

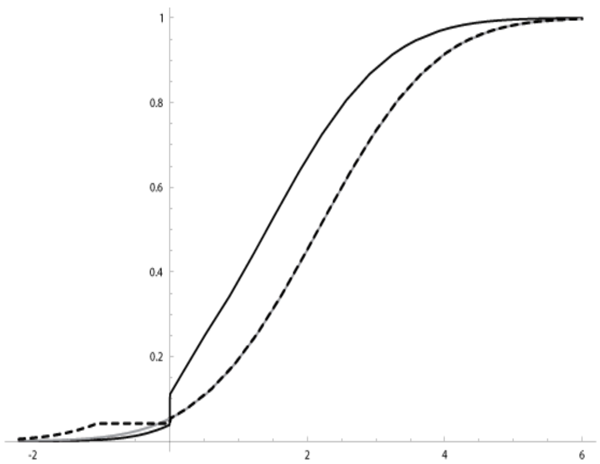

Figure 3.2: Distribution function of the optimal contingent claim for a stock price driven by a geometric Brownian motion. Black line: with UBSR constraint; gray line: without risk constraint; dashed line: with VaR constraint.

where $\bar{x}$ has to be determined such that $Q_1(dP/dQ_0 > \bar{x}) = \alpha$, and λ^* is chosen such that the budget constraint is satisfied.

If there is no risk constraint, then the solution is given by

$$I\left(\tilde{\lambda}\frac{dP}{dQ_0}\right) = -\log(\tilde{\lambda}) + \int_0^T \alpha_s^0 dB_s^0 + \frac{1}{2}\int_0^T (\alpha_s^0)^2 ds,$$

where again $\tilde{\lambda}$ has to be chosen such that the budget constraint is satisfied.

For $\bar{x}_\ell = 0$, $a^0 \equiv 0.3$, $a^1 \equiv 0.2$, $T = 20$, $x_1 = 0.18$, $x_0 = 0.36$, and a VaR-level α at 0.1 under Q_1,[3] Figures 3.2 and 3.3 show the cumulative distribution functions and densities of the optimal contingent claim under different constraints under the measure Q_0. We compare the solution X^* of the maximization problem under the UBSR constraint (black line) with the solutions of the problem without risk constraint (gray line) and with a VaR constraint (dashed line). Both risk constraints limit the probability of losses considerably. The VaR constraint, however, leads to a higher probability of very large losses compared to the solution without any risk constraint. In this case there is only a slight change in the distribution of the VaR-optimal contingent claim for positive values, the main shift takes place on the negative side where the probability of small losses is decreased whereas

[3]Observe that VaR is calculated under the measure Q_1 while the distribution and density functions are plotted under the measure Q_0.

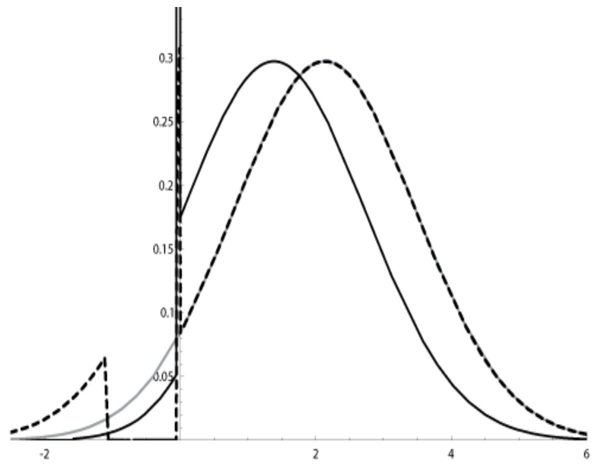

Figure 3.3: Density function of the optimal contingent claim for a stock price driven by a geometric Brownian motion. Black line: with UBSR constraint; gray line: without risk constraint; dashed line: with VaR constraint.

the probability of very large losses is increased. Risk management based on VaR encourages insurance against medium size losses, but favors high losses. UBSR, in contrast, also reduces the risk of very high losses. Regulators and managers should hence better use UBSR measures instead of VaR in order to prevent high losses.

3.4.2 A Pure Jump Model

In the second example we will investigate what happens if the stock price is driven by a pure jump process instead of a geometric Brownian motion. We restrict our attention to a stock price which is driven by a Poisson process $N = (N_t)_{0 \leq t \leq T}$ with jump rate λ under the measure Q_0. We assume that N generates the filtration. The process M defined by by $M_t^0 := N_t - \lambda t$ $(0 \leq t \leq T)$ is a Q_0-martingale. We assume that the stock price S is a geometric Poisson process whose dynamics can be described by the following stochastic differential equation,

$$dS_t = \mu^0 S_t dt + \sigma S_{t-} dM_t^0 \qquad (0 \leq t \leq T)$$

for some $\mu^0 \in \mathbb{R}$ and $\sigma > 0$ such that $\mu^0/\sigma < \lambda$. Then the financial market is complete, and the unique absolutely continuous and equivalent martingale measure is given by the Radon-Nikodym density

$$\frac{dP}{dQ_0} = \exp(\alpha^0 T) \left(1 - \frac{\alpha^0}{\lambda}\right)^{N_T},$$

where $\alpha^0 := \mu^0/\sigma$.

We assume that the subjective probability measure Q_1 is specified in the following way. Let $\mu^1 \in \mathbb{R}$ be given such that $\mu^1/\sigma < \lambda$. With $\alpha^1 := \mu^1/\sigma$, we let Q_1 be the measure under which M^1 with $M_t^1 := M_t^0 + (\alpha^0 - \alpha^1)t$ $(0 \leq t \leq T)$ is a martingale. Then the density of Q_1 with respect to Q_0 is given by

$$\frac{dQ_1}{dQ_0} = \exp((\alpha^0 - \alpha^1)T) \left(1 - \frac{\alpha^0 - \alpha^1}{\lambda}\right)^{N_T}.$$

The dynamics of the stock price can be rewritten in terms of M^1:

$$dS_t = \mu^1 S_t dt + \sigma S_{t-} dM_t^1 \qquad (0 \leq t \leq T).$$

Letting

$$\begin{aligned}
A &:= \left\{\lambda_2^* \frac{dP}{dQ_0} > e^{-\bar{x}_\ell} + \lambda_1^* \frac{dQ_1}{dQ_0} e^{-\bar{x}_\ell}\right\}, \\
B &:= \left\{e^{-\bar{x}_\ell} \leq \lambda_2^* \frac{dP}{dQ_0} \leq e^{-\bar{x}_\ell} + \lambda_1^* \frac{dQ_1}{dQ_0} e^{-\bar{x}_\ell}\right\}, \\
C &:= \left\{\lambda_2^* \frac{dP}{dQ_0} < e^{-\bar{x}_\ell}\right\},
\end{aligned}$$

the optimal contingent claim is given by

$$\begin{aligned}
X^* &= x^*\left(\lambda_1^* \frac{dQ_1}{dQ_0}, \lambda_2^* \frac{dP}{dQ_0}\right) \\
&= \Bigg[-\log(\lambda_2^*) - a^0 T - N_T \log\left(1 - \frac{a^0}{\lambda}\right) \\
&\qquad + \log\left(1 + \lambda_1^* \exp\left((a^0 - a^1)T\right)\left(1 - \frac{a^0 - a^1}{\lambda}\right)^{N_T}\right)\Bigg] \cdot 1_A \\
&\quad + \bar{x}_\ell \cdot 1_B \\
&\quad + \left[-\log(\lambda_2^*) - a^0 T - N_T \log\left(1 - \frac{a^0}{\lambda}\right)\right] \cdot 1_C,
\end{aligned}$$

where λ_1^* and λ_2^* have to be chosen such that X^* satisfies the budget constraint.

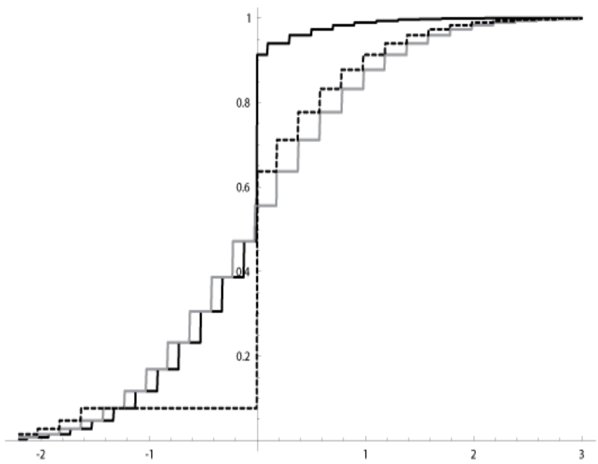

Figure 3.4: Distribution function of the contingent claim for a stock price driven by a pure jump process. Black line: with UBSR constraint, gray line: without risk constraint, dashed line: with VaR constraint

Under the VaR constraint $Q_1(X < \bar{x}_\ell) < \alpha$ the solution is obtained as

$$X^{VaR} = \begin{cases} \bar{x}_\ell & \text{on } \left\{ e^{-\bar{x}_\ell} \leq \lambda^* \frac{dP}{dQ_0} \leq \lambda^* \bar{x} \right\}, \\ -\log(\lambda^*) - a^0 T - N_T \log\left(1 - \frac{a^0}{\lambda}\right) & \text{on the remaining space,} \end{cases}$$

where $\bar{x}$ has to be determined such that $Q_1(dP/dQ_0 > \bar{x}) = \alpha$, and λ^* is chosen such that the budget constraint is satisfied.

If there is no risk constraint, then the solution is given by

$$I\left(\tilde{\lambda}\frac{dP}{dQ_0}\right) = -\log(\tilde{\lambda}) - a^0 T - N_T \log\left(1 - \frac{a^0}{\lambda}\right),$$

where again $\tilde{\lambda}$ has to be chosen such that the budget constraint is satisfied.

For $\bar{x}_\ell = 0$, $a^0 = a^1 \equiv 0.2$, $T = 20$, $x_1 = 0.6$, $x_0 = -0.9$, and a VaR-level α at 0.08, Figure 3.4 shows the cumulative distribution functions for the optimal solutions under different constraints under the measure Q_0. We compare the solution X^* of the maximization problem under the UBSR constraint (black line) with the solutions of the problem without risk constraint (gray line) and with a VaR constraint (dashed line). The results in the case of a pure jump stock price resemble the effects which we have already observed in the continuous model above. However, the jump of the dashed line in zero is much larger here than in the previous example since here the VaR constraint is strong enough to also shift mass from the negative to the positive side.

3.5 Conclusion

We provide a solution to the utility maximization problem (3.15) under a joint budget and UBSR constraint without model uncertainty in a "complete market" setting. We characterize precisely under which conditions the budget constraint is too strict and no solution can be obtained. Otherwise, there exists a solution to the maximization problem. In the latter case, the solution is explicitly determined. The derivation requires a careful analysis of the constraints.

We then consider the utility maximization problem (3.18) under our two constraints in the presence of model uncertainty in an incomplete market model. Here we assume that the utility function is only finite on the positive halfline. We characterize the measures that solve the dual problem as certain worst case measures for the optimal contingent claim. Then we give the solution to the utility maximization problem using this characterization.

Furthermore, we compare our solution to two benchmark portfolios: the optimal solutions of the utility maximization problems without risk constraint, and with a VaR constraint. This example illustrates that a regulator or manager should favor the UBSR constraint over a VaR constraint. A VaR constraint leads to large losses in the worst states. Compared to both a VaR constraint and no risk constraint, the UBSR constraint decreases the size of the losses considerably. Thus, the convex risk measure UBSR is not only superior to VaR from the perspective of the axiomatic theory of risk measures, but also influences investments of rational agents in a desirable way.

If one wants to solve the utility maximization problem under our two constraints also for utility functions that are finite on the whole real line, there occurs an additional difficulty: As in Chapter 2, we have to work with a subset $\mathcal{P}_\ell$ of the absolutely continuous martingale measures when we solve the loss minimization problem, and with another subset $\mathcal{P}_u$ for the utility maximization problem. These two sets may not coincide, and then we cannot distinguish the four different cases of Theorem 3.3.13 as clearly any more.

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Index of Notations

$(\Omega, \mathcal{F})$	a measurable space
$(\mathcal{F}_t)_{0\le t\le T}$	a filtration with $\mathcal{F}_t = \mathcal{F}$
$(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{0\le t\le\infty})$	the enlarged space; see Section 1.2
$\mathcal{M}_1(\Omega)$	the set of probability measures on Ω
R	a reference measures
S	a semimartingale in Section 1.2, the stock price process in Chapters 2 and 3
$\mathcal{Q}$	a subset of $\mathcal{M}_1(\Omega)$ in Chapter 1, the set of subjective measures in Chapter 2
$\mathcal{Q}_0,\ \mathcal{Q}_1$	two sets of subjective measures in Chapter 3
$\mathcal{P}$	a subset of $\mathcal{M}_1(\Omega)$ in Section 1.1, the set absolutely continuous martingale measures in Section 1.2 and Chapter 2
$\mathcal{P}_e$	the set of equivalent martingale measures
$\bar{\mathcal{P}}$	the set of extended martingale measures; see Definition 1.2.3
$\mathcal{P}^T$	the set of projections on $(\Omega, \mathcal{F})$ of extended martingale measures; see Definition 1.2.3
$\mathcal{P}_0$	a subset of $\mathcal{P}$
$\mathcal{P}'$	$\mathcal{P}$ if $\bar{x}_u = -\infty$, and $\mathcal{P}^T$ if $\bar{x}_u = 0$; see (2.3) and (2.4)
$\mathcal{P}'_0$	$\mathcal{P}_0$ if $\bar{x}_u = -\infty$ and $\mathcal{P}^T$ if $\bar{x}_u = 0$
$\mathcal{C}_f$	the set of measures (P, Q_1, Q_0) with finite generalized divergence defined in (3.65)
$\mathcal{K}_{\mathcal{P}}$	the set of densities of measures in $\mathcal{P}$ with respect to R; see Assumption 1.1.1
$\mathcal{K}_{\mathcal{Q}}$	the set of densities of measures in $\mathcal{Q}$ with respect to R; see Assumption 1.1.1
δ_x	the Dirac measure in x
P^a, P^s	the absolutely continuous and singular part of P with

	respect to some given measure
Q^t	the projection of a measure $\bar{Q}$ on $(\bar{\Omega}, \bar{\mathcal{F}})$ to $(\Omega, \mathcal{F}_t)$
$\frac{dP}{dQ}$	the generalized Radon-Nikodym derivative; see (2.18)
$E_Q[X]$	the expectation of the random variable X with respect to the measure Q
$E_Q[X; A]$	the expectation of X with respect to Q given $A \in \mathcal{F}$, also written as $E_Q[X \cdot 1_A]$
1_A	the indicator function on the set A
f	a convex function on $[0, \infty)$
$f(\cdot, \cdot)$	the corresponding convex function on $[0, \infty)^2$; see (1.1)
$\hat{f}$	$xf(1/x)$
u	a concave utility function; see Section 2.1
I	$(u')^{-1}$
v	the convex conjugate of u; see (2.14)
v_λ	$v_\lambda(x) := v(\lambda x)$
$v(\cdot, \cdot)$	the corresponding function on $[0, \infty)^2$; see (2.15)
ℓ	a convex loss function; see Section 3.1.1
L	the generalized inverse of ℓ'; see (3.22)
x^*	the solution of a deterministic maximization problem; see (3.20)
$v(\cdot, \cdot, \cdot)$	the convex function in the dual problem in Chapter 3; see Lemma 3.2.12(x)
$\tilde{v}(\cdot, \cdot)$	the convex function in the dual problem of the loss minimization problem in Chapter 3; see Lemma 3.2.12(xi)
$f(P\|Q)$	the f-divergence of P with respect to Q; see Definition 1.0.1
$f(\mathcal{P}\|\mathcal{Q})$	$\inf_{P \in \mathcal{P}} \inf_{Q \in \mathcal{Q}} f(P\|Q)$
$F_R(\phi, \psi)$	the convex functional defined in (1.6)
$H(P\|Q)$	the relative entropy of P with respect to Q: $E_Q\left[\frac{dP}{dQ} \log\left(\frac{dP}{dQ}\right)\right]$ if $P \ll Q$
$v_\lambda(P\|Q)$	the v_λ-divergence of P with respect to Q; see Definition 2.19
$v_{\lambda_1,\lambda_2}(P\|Q_1\|Q_0)$	the convex functional defined in (3.33)
$\tilde{v}_c(P\|Q_1)$	the $\tilde{v}_c$-divergence defined in (3.34)
$\bar{x}_u$	the left boundary of the domain of the utility function

	u; see Section 2.1
$\bar{x}_\ell$	the point where the loss function ℓ hits zero; see Section 3.1.1
x_0	the initial endowment in Chapters 2 and 3
x_1	the risk limit in Chapter 3
y_0	the minimum level of expected utility in Section 2.4.3
$\mathcal{V}(x_0)$	the set of portfolio value processes defined in (1.18)
$\bar{\mathcal{V}}(x_0)$	the set of corresponding portfolio processes on $(\bar{\Omega}, \bar{\mathcal{F}}, (\bar{\mathcal{F}}_t)_{0\le t\le\infty})$ defined before Definition 1.2.3
$\mathcal{X}_{P,Q}(x_0)$	the set of affordable contingent claims defined in (2.21) in the simplified problem of Section 2.2
$\mathcal{X}(x_0)$	the set of affordable contingent claims defined in Section 2.3.2
$\mathcal{X}_{P,Q_1,Q_0}(x_0, x_1)$	the set of admissible contingent claims defined in (3.16)
$\mathcal{X}(x_0, x_1)$	the set of admissible contingent claims defined in (3.19)
$L^1(R)$	the space of random variables on $(\Omega, \mathcal{F}, R)$ with $E_R[\lvert X\rvert] < \infty$
$L^1_+(R)$	the subspace of non-negative elements in $L^1(R)$
$L^h(Q)$	an Orlizc space; see (1.9)
$\lvert\lvert X\rvert\rvert_{Q,h}$	an Orlizc norm; see (1.10)
$P \ll Q$	P is absolutely continuous with respect to Q
$P \sim Q$	P is equivalent to Q
$x \vee y$	the maximum of x and y
$x \wedge y$	the minimum of x and y

Selbständigkeitserklärung

Hiermit erkläre ich, dass ich die Arbeit selbständig und nur unter Verwendung der angegebenen Hilfsmittel und Hilfen angefertigt habe.

Berlin, den 2. Februar 2006